Portland State University PDXScholar

Dissertations and Theses

Dissertations and Theses

1-3-2019

Robust Maximum Flow Network Interdiction Problem

Darshan Rajesh Chauhan Portland State University

Follow this and additional works at: https://pdxscholar.library.pdx.edu/open_access_etds

Part of the Civil and Environmental Engineering Commons Let us know how access to this document benefits you.

Recommended Citation

Chauhan, Darshan Rajesh, "Robust Maximum Flow Network Interdiction Problem" (2019). *Dissertations and Theses.* Paper 5442.

This Thesis is brought to you for free and open access. It has been accepted for inclusion in Dissertations and Theses by an authorized administrator of PDXScholar. For more information, please contact pdxscholar@pdx.edu.

Robust Maximum Flow Network Interdiction Problem

by

Darshan Rajesh Chauhan

A thesis submitted in the partial fulfillment of the requirements for the degree of

Master of Science in Civil and Environmental Engineering

> Thesis Committee: Avinash Unnikrishnan, Chair Stephen Boyles Miguel A. Figliozzi Thomas Schumacher

Portland State University 2019

© 2019 Darshan Rajesh Chauhan

Abstract

In this thesis, a maximum flow-based network interdiction problem considering uncertainties in arc capacities and interdiction resource consumption is solved. The problem consists of two entities with opposing objectives: the goal of the adversary is to maximize the flow of illicit drugs through the network, while the goal of the interdictor is to minimize the maximum flow by completely interdicting arcs given a specified amount of resources. Lack of complete information about the usage patterns of the transportation network by the adversary results in an uncertain estimate of arc capacity and resources required for interdiction by the interdictor. To account for this uncertainty, a robust optimization framework is utilized, resulting in a Robust Network Interdiction Problem (RNIP).

A novel mixed-integer linear program is proposed that solves the RNIP. Three heuristics are proposed to solve RNIP, the first based on Lagrangian Relaxation, the second based on Benders' Decomposition, and the third based on Benders' Decomposition enhanced using the Lagrangian Relaxation presolve. Computational experiments show that the third heuristic performs the best with a final MIP gap of less than 5% and a computational time saving of more than 90% for all the test networks when compared to a state-of-the-art mixed integer program solver. Sensitivity analyses are performed to identify budgets of uncertainty that provide a realistic estimate of the actual maximum flow using a Monte Carlo simulation scheme. Finally, robust decisions are compared to decisions not accounting for any uncertainty to evaluate the *value of robustness*. It is found that robust decisions can provide fairly accurate estimates of possible actual maximum flows in the network. When the interdiction efforts are significant, robust decisions also lead to a reduction in actual maximum flows, as much as 78% on average for a series of test networks, when compared to decisions not accounting for any uncertainty. To my family

Acknowledgments

This work is a product of the best wishes and support of a lot of people. I will try and do my best to acknowledge everyone.

Foremost, I would like to sincerely thank Prof. Avinash Unnikrishnan, who has been a mentor in the truest sense. Through the last two years, I have grown a lot personally and academically, and Prof. Unnikrishnan gets most of the credit for this. I would like to especially thank him for his patience, continuous motivation, and our countless discussions about this thesis, research, and life in general.

I would also like to thank Prof. Stephen Boyles, Prof. Miguel Figliozzi, and Prof. Thomas Schumacher for agreeing to serve on my thesis committee and adding value to this work.

I am very grateful to the National Science Foundation (NSF) for enabling this work through much needed financial support. This research is a product of NSF grants awarded to Prof. Unnikrishnan and Prof. Boyles (grant numbers: CMMI-1562109/1562291 and CMMI-1254921). I am also thankful for the support I received through the Maseeh Fellowship (by the Massiah Foundation through Portland State University) and the Walter H. Kramer Fellowship (by the Transportation Research and Education Center at Portland State University). I would like to thank Dr. Kalyana Rama J. S. for introducing me to the field of research during my undergraduate studies. I cannot thank Dr. Arkamitra Kar enough for pushing over the ledge for pursuing graduate studies, for everything he has taught me, and for our friendship which I truly appreciate.

My time in Portland was enriched by a lot of people. I would like to thank my lab-mates Santiago, Travis, Rohan, Jael, Joe, Jaclyn, Katherine, Frank, Jason, David, Mostafa, and Ali for sharing recent news and ideas about transportation, politics, and life stories from the U.S. or back home (a lot of it over pizza or games of table tennis!). Thanks to Sam, Kiley, Megan, and Ariel for not just being great staff but also friends and helping me through all the administrative hurdles. Thanks to Abhishek and Bikesh for all of our weekend trips, ramen, and boba tea runs, and other random stuff that made me look forward to Saturdays. My talks with Arjun over long walks are something I treasure. Monish's three vacations to Portland and our trips to Seattle and Washington D.C. over the last year will always be cherished.

A big shout out to my family: *baa*, mom, papa, *mota* mom, *mota* papa, Parth, Vaibhav, Bhumika, Chintan, and Piyu, who always believed in me, supported me, and made sure I was doing fine all through my journey here over my daily calls to India. My love for my long-distance friends Monish, Aditya, Amod, Shrineh, Siddhi, Raghavi, Krisha, Anvita, Nilay, Mitul, Jaideep, and Shashank for being the most awesome support system and so much more.

V

To all other people whom I could not acknowledge here, thanks for being a part of my adventure.

Contents

	Abstract	i
	Dedication	iii
	Acknowledgments	iv
	List of Tables	x
	List of Figures	xi
1	Introduction	1
	1.1 Background and Motivation	1
	1.2 Applications of Network Interdiction Models	7
	1.3 Problem Statement and Objectives	9
	1.4 Organization	12
2	Literature Review	14
	2.1 Network Interdiction	14
	2.2 Solution Methodologies for network interdiction	21
	2.3 Robust Optimization	22
3	Problem Description and Formulation	28
	3.1 Problem Description	28

	3.2	Model Assumptions	30
	3.3	Nomenclature	31
	3.4	Formulation	33
		3.4.1 Dualizing in two stages	35
		3.4.2 Realizing the deterministic problem and adding robustness	39
	3.5	Check for correctness of the robust formulation	47
	3.6	Extensions	49
		3.6.1 Multiple resources required for interdiction	49
		3.6.2 Interdiction at various levels with different economies of scale	50
	3.7	Summary	53
4	Solı	tion Methodology 5	54
4	Solu 4.1	tion Methodology State	54 54
4	Solu 4.1 4.2	tion Methodology State S	54 54 59
4	Solu 4.1 4.2 4.3	tion Methodology tion Valid Upper Bounds text Robust Knapsack Problem text Lagrangian Relaxation text	54 54 59 62
4	Solu4.14.24.3	tion Methodology \$ Valid Upper Bounds Robust Knapsack Problem Lagrangian Relaxation 4.3.1 Solution Procedure	54 54 59 62 67
4	Solu 4.1 4.2 4.3	tion Methodology tion Valid Upper Bounds text Robust Knapsack Problem text Lagrangian Relaxation text 4.3.1 Solution Procedure 4.3.2 Lagrangian Parameters' Updating Procedure	54 59 62 67 72
4	 Solu 4.1 4.2 4.3 4.4 	tion Methodology Image: Second Se	54 59 62 67 72 73
4	 Solu 4.1 4.2 4.3 4.4 	tion Methodology \$ Valid Upper Bounds \$ Robust Knapsack Problem \$ Lagrangian Relaxation \$ 4.3.1 Solution Procedure 4.3.2 Lagrangian Parameters' Updating Procedure \$ Benders' Decomposition-based Heuristic \$ 4.4.1 A Simultaneous Penalty Heuristic for solving the Master Problem \$	 54 54 59 62 67 72 73 75
4	 Solu 4.1 4.2 4.3 4.4 	tion Methodology a Valid Upper Bounds a Robust Knapsack Problem a Lagrangian Relaxation a 4.3.1 Solution Procedure 4.3.2 Lagrangian Parameters' Updating Procedure a Benders' Decomposition-based Heuristic a 4.4.1 A Simultaneous Penalty Heuristic for solving the Master a Problem a 4.4.2 Solution procedure a	54 54 59 62 67 72 73 75 80

	4.6	Summary	85
5	Ana	alysis	87
	5.1	Computational Efficiency	89
	5.2	Sensitivity to Changes in Amount of Uncertainty in Parameters	97
	5.3	Sensitivity to Changes in Budgets of Uncertainty	103
	5.4	Value of Adding Robustness	108
	5.5	Summary	120
6	Cor	nclusions	123
	6.1	Summary and Conclusions	123
	6.2	Directions for future research	126
	Bib	liography	128

List of Tables

1.1	Summary of scenarios considered in the motivating example 7
5.1	Test Networks
5.2	Performance of Gurobi and heuristics for RNIP
5.3	Sensitivity to changes in the amount of uncertainty of parameters 102
5.4	Sensitivity to changes in the budget of capacity uncertainty 104
5.5	Sensitivity to changes in the budget of resource consumption uncer-
	tainty
5.6	Value of adding robustness in decision-making ($\Delta = 2000$)
5.7	Value of adding robustness in decision-making ($\Delta = 10000$) 116
5.8	Value of adding robustness in decision-making ($\Delta = 15000$) 119

List of Figures

1.1	Motivating capacity and resource consumption uncertainty 5
3.1	Scaling of cumulative costs with respect to total output
5.1	Visual representation of 4×4
5.2	Savings in overall computational time compared to Gurobi 91
5.3	Maximum final gap values while solving RNIP
5.4	Maximum deviation from Gurobi upper bound solution 94
5.5	Quality of the first upper bound determined by heuristics 97
5.6	Minimum cut and the interdiction decisions of a $2{\times}2$ test network 109
5.7	Sensitivity to changes in Γ ($\Pi = 2, \Delta = 2000$)
5.8	Sensitivity to changes in Π ($\Gamma = 20, \Delta = 2000$)
5.9	Value of adding robustness ($\Delta = 10000$)
5.10	Value of adding robustness ($\Delta = 15000$)

1 Introduction

1.1 Background and Motivation

Network-based structures are ubiquitous. They exist as transportation networks, water supply and distribution networks, energy generation, and distribution networks, telecommunication networks, biological networks, computer networks, the internet, etc. Moreover, there are problems where the network-based structure is not apparent, but they can be visualized and formulated as network-based problems. Examples of such problems include creating a schedule, allocation of inspection resources, matrix rounding problems, and many more Ahuja et al. (1993)]. The pervasiveness of these network-based structures makes it important to identify and study the vulnerable connections of a network. The identification of the most vulnerable arcs (also, links) of a network is a way to evaluate the significance of their availability and has applications in different kinds of scenarios. Some examples include controlling the spread of an infection [Assimakopoulos (1987), interdiction of illicit drugs [Steinrauf (1991)], interdiction of enemy resources in a war [Loh (1991)], prioritizing infrastructure improvements [Lu et al. (2014)], preparing against terrorism threats [Ayvildiz et al. (2018)], etc. Moreover, uncertainties are innate in transportation networks, stemming from human factors like varied perception of the transportation network due to differences in

psychological and behavioral attitudes [Arentze and Timmermans (2005)], or built environment features [Magliocca et al. (2019)], or weather conditions [Lam et al. (2008), Sumalee et al. (2011)], or other random causes like crashes, making the identification of the most important arcs more challenging. The overarching goal of this work is the identification of the vulnerable connections in a network under uncertainty.

Consider the problem of drug interdiction as an example. Illicit drugs and the associated criminal organizations pose a significant threat to national security, public health, and law enforcement [DEA (2018)]. The increasing presence of illicit drugs is indicated by the increasing amount of drug seizures by local enforcement agencies, the increasing usage among the populations, and the increasing amount of mortalities related to drug overdose, in the U.S. and around the world [DEA (2018), UNO (2018)]. Identifying the most important arcs of the drug flow network and destroying them would result in a maximum reduction of drug flow at the demand point in the most efficient manner. However, the intelligence agencies can still only speculate about the criminal organization's (the adversary) perception and resulting behavior towards the risk of interdiction. The current work is a small step towards helping alleviate this problem by modeling the decision-making process involved in the interdiction of illicit drugs (the commodity), from the perspective of a local enforcement agency (the interdictor), considering uncertainties in arc capacities and resource consumption for interdiction.

To account for the uncertainty, the decision-maker can either opt for stochastic optimization or robust optimization. In stochastic optimization, the uncertainty is described by a probability distribution, while in robust optimization, the uncertainty set is deterministic and set-based [Bertsimas et al. (2011)]. In the stochastic optimization framework, it is often cumbersome to enumerate all possible scenarios and assign them appropriate weights, or it may not be possible at all due to insufficient data on the exact variation (i.e. probability distribution) of parameters involved in the problem. The robust optimization approach benefits from a low amount of data requirements and independence from the knowledge of the probability distribution. Only the bounds on the variation of parameters may be sufficient for the decision-making process in some cases. As the amount of information on the transportation network of illicit drugs is sparse, it is easier to estimate the bounds of the network parameters than their exact probability distributions. Therefore, the robust optimization approach is used in the current study. So as to not reinvent the wheel again, interested readers are referred to Kall et al. (1994), Nemirovski and Shapiro (2006), and Shapiro et al. (2009) to learn about various methods in stochastic optimization and programming. The work done by Ben-Tal et al. (2009), Bertsimas et al. (2011), and Gabrel et al. (2014) serves as a great reference on the theory and applications of robust optimization. As robust optimization does not assume any probability information about the uncertainty,

a whole field of robust optimization which is dedicated to addressing robustness across different probability distributions. To learn more about the growing field of distributionally robust optimization refer to Delage and Ye (2010), Goh and Sim (2010), and Wiesemann et al. (2014).

To understand the role of uncertainty, consider the example transportation network shown in figure 1.1. Node s refers to the source of the commodity and node t refers to the destination market for the commodity. The network parameter u refers to the nominal estimated capacity of an arc, and r refers to the nominal amount of resource required to destroy the arc completely. \bar{u} refers to the estimated worst-case capacity of the arc, and \bar{r} refers to the worst-case amount of resources required to completely destroy the arc. There are three paths, A, B, and C, which the adversary uses, and their goal is to maximize the flow of the commodity to the destination t. On the other hand, the goal of the interdictor is to minimize the worst-case availability of drugs at destination t subject to the availability of limited resources. In the current example, the interdictor has 14 units of resource available.



Figure 1.1: Motivating capacity and resource consumption uncertainty

If the interdictor does not consider uncertainty, the decision will be based solely on the nominal values available. In the current case, the interdictor would decide to interdict paths B and C using 11 units of interdiction resource and the availability of the commodity at destination t would be 100 units in the nominal case and 150 units in the worst-case. Now, consider one of the simplest models for robust optimization in the decision-making process, the worst-case hedge. In the worst-case hedge model, the decision-maker chooses worst-case values of parameters to ensure that the solution protects against all uncertain scenarios. For the worst-case hedge model, the decision will be based on the worst-case values of capacity and resource consumption (i.e. \bar{u} and \bar{r}). In this case, the interdictor would decide to interdict paths A and C using all the 14 units of interdiction resource, and the availability of the commodity at the destination t would be 120 units in the nominal case and 130 units in the worst-case. Notice that the interdictor is compromising by allowing 20 extra units of the commodity in the nominal case to reduce the worst-case availability by 20 units.

As the size of the network grows, being totally risk-aversive (i.e. the worstcase hedge model) would lead to a huge increase in the nominal availability of drugs at the destination, while there will also be a significant reduction in the worst-case availability of drugs. The bump in the nominal value, as the network size grows, may not seem like a reasonable compromise, considering the shrinking likelihood of every network parameter being at their worst simultaneously. What if a better compromise could be reached by controlling the amount of uncertainty that is incorporated into modeling? The answer to this is the concept of budgeted uncertainty, wherein additional constraints are enforced to moderate the amount of uncertainty considered while modeling.

Consider the network example shown in figure 1.1 again. Now, additional constraints are enforced so that the interdictor would consider worst-case arc capacity and resource consumption for interdiction at only one path each (can be different paths). The updated decision of the interdictor would be to interdict paths A and B using all of 14 units of resources (worst-case value for path A, nominal value for path B). The availability of the commodity at the destination t would be 110 units in the nominal case and 140 units in the worst case. The summary of the results of drug-availability at the destination t obtained so far is given in table 1.1.

6

	Nominal case	Worst-case
No uncertainty considered	100	150
Worst-case hedge	120	130
Robust optimization with budgeted uncertainty	110	140

Table 1.1: Summary of scenarios considered in the motivating example

In table 1.1, it can be noticed that the case of robust optimization with budgeted uncertainty provides the interdictor with a new compromise which does not increase the nominal availability of the commodity at destination t as much as the worst-case hedge approach but provides worst-case protection better than the case where no uncertainty is considered. The case of budgeted uncertainty, therefore, provides the decision-maker an avenue to calibrate the amount of risk undertaken and not be overly pessimistic, while still enjoying the perks of a risk-aversive approach.

1.2 Applications of Network Interdiction Models

Network interdiction models have garnered a great amount of attention due to its applicability in a wide range of areas, and also because of its captivating combinatorial nature. This section elaborates on the applications of network interdiction problems to further motivate the problem. The first domain of their applications is to problems related to national security like: warfare [Wollmer (1964), McMasters and Mustin (1970)]; protecting critical infrastructure [Brown et al. (2006), Yates and Lakshmanan (2011)]; preparing against terrorism threats [Ayyildiz et al. (2018)]; intercepting drugs, weapons, and nuclear smuggling [Wood (1993), Pan et al. (2003), Morton et al. (2007), Nehme and Morton (2009), Malaviya et al. (2012), Sullivan et al. (2014), Enayaty-Ahangar et al. (2019)]; and protecting electrical power grids from terrorists [Salmeron et al. (2004), Motto et al. (2005), Salmeron et al. (2009), Arroyo (2010), Ding et al. (2018)].

Network interdiction models are also employed for humanitarian purposes and improving public health like: protecting electrical power grids from disasters[Yuan et al. (2016)]; flood control [Ratliff et al. (1975)]; control in infection spread [Assimakopoulos (1987), Nandi and Medal (2016)]; chemical treatment of toxic waste [Phillips (1993)]; impact of natural disasters on fiber-optic networks [Neumayer et al. (2015), Neumayer et al. (2015)]; protecting water distribution systems [Perelman and Amin (2014)]; and routing hazardous materials through the network [Yates (2013), Kheirkhah et al. (2016)].

Interdiction models can also be used to inform policies at corporations and local government like: prioritization of infrastructure improvements [Lu et al. (2014)]; policy for toll control [Borndörfer et al. (2016)]; identifying and protecting vulnerable facilities [Church et al. (2004), Church and Scaparra (2007), Scaparra and Church (2008), Lei (2013)]; vulnerability assessment of supply chains [Scaparra and Church (2008), Gedik et al. (2014)]; monitoring of computer

8

networks [Smith and Lim (2008)]; impact of denial-of-service (DoS) attacks on a network [Fu and Modiano (2019)]; and designing strategies for enhancing security in cyber-physical systems [Sanjab et al. (2017)].

1.3 Problem Statement and Objectives

This work focuses on modeling interdiction in a maximum flow network while considering uncertainties in the arc capacities and uncertainties in the consumption of resources required to interdict an arc. Consider the example of drug interdiction for motivating the presence of uncertainty arc capacity and resource consumption. For drug interdiction in a maximum flow network, each arc has two capacities: first, the physical capacity which is limited by available network infrastructure (like, numbers of lanes in the road (or the width and depth of a river), quality of the road (or the amount of obstacles in a river)); and second, the perception of capacity in the mind of the enemy (or, adversary) which is influenced by the amount of surveillance and regular patrolling done by the local enforcement agency. While the prior is relatively easier to determine exactly with high certainty, the latter is not. Also, in the cases where the second capacity is the limiting factor, a significant amount of capacity uncertainty can be observed. The uncertainty in resource consumption stems from the variation in the number of protection forces sent by the enemy with the drug flow, and dependence of interdiction resources on weather conditions, etc.

It is assumed that the adversary has a practically infinite amount of commodity, money, and other resources available. The primary bottleneck faced by them is moving their commodity across the network. The objective of the adversary is to maximize the commodity availability at the destination by maximizing the flow through the network. Therefore, the objective of the interdictor would be to minimize the maximum flow of commodity through the network subject to the total resources available for interdiction. Similar assumptions have been made in several previous network interdiction studies like Wood (1993) and Cormican et al. (1998).

It is assumed that whenever an arc is interdicted, it is destroyed completely and immediately, i.e. the capacity drops to zero. The enemy forces can build the arc back up but that would require time much greater than the planning period of interdiction.

Currently, it is assumed that a single kind of commodity flows through the network from a single source to a single sink. However, the problem can be extended to account for multiple sources and sinks, and special cases of multiple commodity flow [Wood (1993)].

It is also assumed that only a single kind of resource is required for the interdiction of the commodity. However, the problem can be extended to accommodate choices among different resources required for interdiction or a combination of resources required for interdiction [Wood (1993)]. The model requires the input of network parameters (capacity and interdiction resource consumption for each arc, nominal and maximum deviation values), the budget of uncertainty for arc capacity variation, the budget of uncertainty for the variation in interdiction resource consumption, and the total amount of interdiction resource available. The outcomes of the model include the arc set for interdiction, and the estimated worst-case maximum commodity flow through the network, along with the minimum capacity cut and its forward flowing arcs.

The major objectives of this thesis are given as:

1. Formulating the robust network interdiction problem as a mixed-integer linear program:

Defining the robust network interdiction problem (RNIP) in the context of illicit drug interdiction as a min-max network flow problem and deriving a mixed-integer linear program for the same.

2. Developing efficient heuristics to solve the RNIP

For this objective, valid upper bounds are derived for the variables of the problem. The 'maximum profit knapsack problem' is revisited in the context of item weight uncertainty. The above results are used to develop three heuristics. The first is a Lagrangian Relaxation procedure to solve the RNIP. The second is a heuristic based on Benders' Decomposition using nominal constraints on capacity and interdiction resource consumption derived by exploring the nature of RNIP. It is shown that the Master Problem of Benders' Decomposition has the same complexity as RNIP, and a simultaneous penalty heuristic is proposed to solve it. The third is a heuristic based on Benders' Decomposition initialized with solution bounds and improved constraints on capacity and interdiction resource consumption found using the Lagrangian Relaxation procedure. All three heuristics also provide information on the MIP gap.

3. Computational analyses

Computational analyses are performed to analyze the performance of the heuristics compared against a state-of-the-art MIP solver. Sensitivity analysis is conducted to find the impact of an increased amount of uncertainty in the network parameters, and the impact of changes in the budget of uncertainty using a Monte Carlo simulation scheme. This is followed by analyzing the value of robustness in the decision-making by comparing robust decisions with decisions not accounting for any uncertainty.

1.4 Organization

The rest of the thesis is organized as follows: Chapter 2 reviews the relevant literature related to network interdiction models, efficient solution methodologies for network interdiction, and robust optimization and its applications to network flow problems. In chapter 3, we model the network interdiction problem considering uncertainties in capacity and resource consumption as a mixed-integer linear program. In Chapter 4, three heuristics to solve the formulated robust network interdiction problem based on Lagrangian Relaxation, and Benders' Decomposition are developed. Chapter 5 examines the performance of the developed heuristics against a state-of-the-art mixed-integer program (MIP) solver and evaluates the value of robustness in the existing problem. Sensitivity analysis is also performed to quantify the impact of uncertainty in parameters, and the impact of budgets of uncertainty to the actual maximum flows of the network. Finally, Chapter 6 summarizes the contributions of this work and puts forth future directions for research.

2 Literature Review

This section reviews the past work done in the area of network interdiction and computationally efficient solution methodologies that exist for them. Later a review of robust optimization and its applications in the context of network flow problems is performed.

2.1 Network Interdiction

The review of the existing literature provides several insights into the network interdiction problem. Interdiction decisions and movement decisions of commodity or people happen over a base network that belongs to the defender. Most of these defender networks can be divided into one of the four types: maximum flow networks, shortest path networks, minimum cost flow networks, and evasion in a network. These are discussed in greater detail in the following paragraphs.

When the base network is a maximum flow network, the objective of one of the players is the maximization of the flow from a source to a sink. Loh (1991) solves single-commodity and multi-commodity network interdiction problem with different kinds of objectives on networks based on maximum flow and minimum cut. One of the objectives is to minimize the maximum flow. Steinrauf (1991) studies the single commodity network interdiction problem in two contexts, one of which is minimizing the maximum possible flow of *coca*, the precursor of cocaine, in a case-study based on Bolivia's road and riverine transportation network. Wood (1993) proves the time-complexity of maximum flow network interdiction problem to be strongly NP-complete (NP: Non-deterministic Polynomial) and describes various variants and stronger formulations to tighten the linear relaxation of the integer formulation using valid inequalities. All of the above works model the "minimize the maximum flow" variant of network interdiction as a bilevel problem in the attacker-defender setting and dualize it to obtain a tractable formulation. The attacker-defender setting refers to a Stackelberg game with two decisions in which the attacker decides to interdict the arcs first and then, the defender responds optimally. Rocco S. and Ramirez-Marquez (2009) propose an evolutionary optimization approach, and Granata et al. (2013) propose a branch-and-price algorithm for solving the deterministic maximum flow network interdiction model. Akgün et al. (2011) extend the maximum flow network interdiction to a multi-terminal version which aims to maximize flow among the pre-specified node groups. Cormican et al. (1998) solve the maximum flow interdiction model in a stochastic context considering uncertainty in the interdiction of arcs. Atamtürk et al. (2019) consider stochastic maximum flow network interdiction problem with uncertainty in arc capacities and model it using mean-risk model. Royset and Wood (2007) formulate a bi-objective maximum flow network

interdiction model to find Pareto-optimal solutions to the objectives of "minimizing the maximum flow" and "minimizing the interdiction cost". Altner et al. (2010) provide path-based valid inequalities and find lower bounds on the integrality gaps for the maximum flow network interdiction model, which is extended by new valid inequalities presented in Afshari Rad and Kakhki (2017). Lunday and Sherali (2010), Zheng and Castañón (2012a), Zheng and Castanón (2012b), and Rad and Kakhki (2013) extend maximum flow network interdiction for dynamic networks.

For the cases when the base network is a shortest path network, the objective of one of the players is to find and traverse on the shortest path from a source to a sink. Israeli and Wood (2002) formulate the shortest path network interdiction model as a bilevel optimization problem in an attacker-defender setting. The bilevel problem is dualized to obtain a tractable formulation. Cappanera and Scaparra (2011) study shortest path interdiction problem with complete fortification. They model it as a trilevel problem, and propose an enumeration scheme and heuristic to solve it. Sadeghi et al. (2017) consider a similar problem but with partial fortification, and proposes a Benders' Decomposition framework to solve it. Wei et al. (2018) consider the shortest path network interdiction problem to optimize interdiction resource consumption while enforcing a threshold on the shortest path. They propose Benders' Decomposition and Lagrangian Relaxation to solve it. Rocco S. and Ramirez-Marquez (2010) develop a bi-objective shortest path network interdiction problem with the objectives of maximizing the length of the shortest path and minimizing the amount of resources used for interdiction. They also develop an evolutionary algorithm to solve the problem wherein they use Monte Carlo simulations to generate strategies, and graph theory to evaluate the goodness of the strategies. Zhang et al. (2018) propose a stochastic shortest path problem in which interdiction success is given by a probability. The model also accounts for multiple sources and sinks. Bayrak and Bailey (2008) extend the work done by Israeli and Wood (2002) and incorporate information asymmetry in the shortest path interdiction. It is formulated as a bilevel program which, upon dualization, results in a mixed-integer nonlinear program (MINLP). The MINLP reformulation is linearized and solved. Borrero et al. (2015) propose a sequential shortest path network interdiction problem. In the problem, the adversary knows the network completely, but the interdictor learns more about the network structure as the game progresses. Song and Shen (2016) consider the shortest path network interdiction with uncertainty in travel costs and uses the chance constraint to model uncertainty. Sefair and Smith (2016), and Xu et al. (2017) consider the shortest path network interdiction over dynamic networks.

For the minimum cost flow problem as a base network, the objective of one of the players is the satisfaction of the demands of all nodes at a minimum cost. Gannon (1989) solves the multi-day minimum cost flow problem subject to interdiction using the dual decomposition method. The problem is formulated as a single level decision-making problem and is solved for sparse and planar graphs. However, their proposed method is also applicable to non-planar and dense graphs. Lim and Smith (2007) formulate a multicommodity minimum cost flow network interdiction problem as a bi-level decision-making problem. The bilevel formulation is set in the attacker-defender context. The discrete interdiction variant is solved using a penalty-based reformulation which is linear. The continuous interdiction variant is shown to be a tougher problem than the discrete version and is solved using an exact partitioning algorithm and a heuristic. Boginski et al. (2009) consider a robust minimum cost flow under uncertainty in arc availability and model it by constraining the conditional value-at-risk.

Loh (1991) also puts forth a probabilistic bilevel framework for evasion in a network. The setting of the problem is as follows: the evader tries to traverse a network undetected. The interdictor aims to interdict an arc by being present at an arc to minimize the probability of undetected evasion knowing well the source and the destination of the evader. This problem is also extended for an interdiction team versus an evader. However, no solution methodology is provided. Morton et al. (2007) solve the same problem in a stochastic context. The stochastic version varies from the deterministic version is only that the information on the destination and the source of the evader is unknown. The problem is solved in a defender-attacker setting. They propose an L-shaped Decomposition method coupled with valid inequalities for the path-based formulation. However, these valid inequalities are only true when at most one interdicted path is traversed on the path. Pan and Morton (2008) generalize valid inequalities proposed in Morton et al. (2007) for the cases where more than one interdicted arcs are encountered on the traversed path. Towle and Luedtke (2018) formulate a more compact version of the formulation proposed in Pan and Morton (2008) which is much more computationally efficient even when solved in an MIP solver. Yates and Lakshmanan (2011) provide a constrained binary knapsack approximation to a similar problem which leads to a significant reduction in the computational times. They also report some success in the magnitude of arc coverage and show that the similarity of patterns in the spatial allocation of resources in the approximation algorithm and exact formulation is statistically significant. Michalopoulos et al. (2015) propose an evasion model with uncertainty in budget availability for interdiction and proposes a tabu search heuristic to solve the problem.

Brown et al. (2006) provide bi-level and trilevel optimization formulations for generalized interdiction models for protecting critical infrastructure. The authors also solve three real-world case studies and conclude with the importance of optimization models for such scenarios, and the importance of solving till optimality. The formulations for the bilevel model are available for both the attacker-defender setting and the defender-attacker setting. In the defender-attacker setting, the defender decides to protect the network with an expectation of a future interdiction first, and then the attacker interdicts the network. The defender-attacker-defender setting was used for the trilevel model. Bertsimas et al. (2016) consider a network interdiction problem in which the interdictor randomizes the interdiction attempts rather than choosing the best strategy, which may be more realistic in cases like protecting critical infrastructure against enemy's attacks. Recently, Magliocca et al. (2019) used agent-based modeling with data over 14 years to model the decision-making of a criminal organization involved in drug trafficking to understand the inadequacies in drug-interdiction operations over time and space.

Though there are a plethora of network interdiction studies performed with varied objectives on a different type of networks, the current work only focuses on the interdiction in a maximum flow network with an objective of minimizing the maximum flows, similar to many of the previous drug interdiction studies. The current study models drug interdiction as a bilevel problem in an attacker-defender setting, portraying a proactive approach to interdiction. The current study also assumes 0-1 interdiction. The arc is destroyed completely when it is interdicted, and there is no residual capacity. This is in line with several previous maximum flow network interdiction models [Wood (1993), Cormican et al. (1998), Bingol (2001)]. In this study, we also provide a formulation to allow discrete partial interdiction. This formulation can also be used to model the economies or diseconomies of scale (or a combination of both) in interdiction. In this work, only single-commodity is assumed to flow through the network.

Refer to Smith and Song (2019) for a recent comprehensive survey on net-

work interdiction models and algorithms.

2.2 Solution Methodologies for network interdiction

Cormican (1995) solves the deterministic and stochastic maximum flow network interdiction problems with Benders' Decomposition based heuristics and reports significant time savings in comparison to the standard branch-and-bound approach. Israeli and Wood (2002) solve the shortest path network interdiction problem using Benders' Decomposition and a Covering Decomposition method, both enhanced with super valid inequalities, and report their success over the branch-and-bound approach. Wood (2010) shows that a generalized network interdiction is a Stackelberg game with two players and two decisions, and can be formulated as a bilevel mixed-integer program with opposing objectives. The work also describes a Benders' Decomposition approach to solve the bi-level network interdiction model. The success of Benders' Decomposition in the context of network interdiction motivates its exploration in the current work.

Bingol (2001) solves a deterministic maximum flow network interdiction model using a Lagrangian Relaxation procedure and reports reasonable success. Uygun (2002) extends the work done by Bingol (2001) by augmenting the Lagrangian Relaxation by a restricted branch-and-bound procedure and improves the solution bounds. The new procedure still cannot guarantee a small optimality gap. Royset and Wood (2007) formulate a bi-objective maximum flow network interdiction problem and solve it using a Lagrangian Relaxation scheme enhanced with cut-enumeration through the branch-and-bound procedure similar to Uygun (2002). The Lagrangian Relaxation procedure benefits from faster computational times than the Benders' Decomposition but cannot guarantee good solutions, while Benders' Decomposition benefits in the terms of solution confidence and quality but the computational experience becomes cumbersome as many subproblems need to be solved [Royset and Wood (2007), Cormican (1995)]. Though the Lagrangian Relaxation based procedure achieved limited success in the aforementioned studies, it is still explored in the current study due to its superiority in computational times.

2.3 Robust Optimization

While mathematical programming is a powerful tool to help model real-life problems, it assumes that all the data required for solving the problem is exactly known. However, uncertainty in data can be present due to some reasons like ignorance (like, how much ore is left in a mine), noise (due to measurement errors, incorrect entries, incomplete data), and unforeseen events (like, future demand for a product to be released) [Rosenhead (2001)]. A "robust" solution reduces the optimality in the nominal case to provide flexibility and some hedging against uncertainty. Therefore, one of the important questions that the decision-maker has to answer is what the acceptable amount of degradation in optimality is to
reduce the risk of the infeasibility of solutions [Greenberg and Morrison (2008)]. There are two major schools of thought regarding how to deal with uncertainty: stochastic optimization and robust optimization. Some of the classical models based on stochastic optimization include mean-risk model [Markowitz (1952), Markowitz (1991), recourse model [Dantzig (1955)], and the chance-constrained model [Charnes et al. (1958)]. Stochastic optimization is based on the assumption that one has information about the probability distribution of the uncertainty. This is not the case when modeling the interdiction of illicit drugs as the amount of information available about the specifics of a drug network is rather scarce. It would be easier to estimate the bounds of variation of the network parameters rather than their probability distributions. Robust optimization allows for accounting uncertainty in a deterministic and set-based manner, and is, therefore, preferred to solve the network interdiction problem to minimize the maximum flow subject to uncertainty in capacity and resource consumption. To learn about various methods in stochastic optimization and programming, interested readers are referred to Kall et al. (1994), Nemirovski and Shapiro (2006), and Shapiro et al. (2009).

Some of the common robust optimization models for accounting uncertainty include worst-case hedge (all uncertain parameters assume worst-case values [Soyster (1973)]), minimax regret (minimizing the maximum regret), and uncertainty sets (ellipsoidal uncertainty set [Ben-Tal and Nemirovski (1999), Ben-Tal

and Nemirovski (2002), polyhedral uncertainty sets [Bertsimas and Sim (2004), Bertsimas and Sim (2003)]). The worst-case hedge model is the simplest to solve, but is overly pessimistic and degrades the optimal values to a large extent to ensure absolute uncertainty. Minimax regret procedure also allows simplicity in calculations but is still a lot pessimistic because of its emphasis on worst-case outcomes and independence with its likelihood of occurrence. The method of ellipsoldal uncertainty sets helps constrain the uncertainty in the model but the robust counterpart is nonlinear even when the nominal problems are linear. Polyhedral uncertainty sets constrain the uncertainty in the model and also has a robust counterpart which is linear if the nominal problem is linear. Therefore, the current study uses polyhedral uncertainty sets for applying robust optimization to the network interdiction problem to minimize the maximum flow subject to uncertainty in capacity and interdiction resource consumption. For a more detailed overview on robust optimization, interested readers are referred to Ben-Tal et al. (2009), Bertsimas et al. (2011), and Gabrel et al. (2014).

Dews and Kozaczka (1981) describe the lessons learned from interdiction efforts of the U.S. Air Force over three wars, and state that the planned interdiction activities were too optimistic. The room for uncertainty is a lot because of the overestimation of the enemy's supply needs and underestimation of the adaptability and flexibility of their transportation systems under attack. Using polyhedral uncertainty sets in robust optimization would result in a risk-averse approach and help with the optimism aspect. However, in the current study, the flexibility of their transportation system from the adversary's perspective is considered only with respect to the capacity of the arcs, and the rest of the transportation system is assumed to be static (i.e. no existing arcs are completely abandoned, no new arcs are generated, the source or the destination of the entire operation do not change, etc.).

Robust optimization using polyhedral uncertainty sets was first introduced by Bertsimas and Sim (2004). The polyhedral sets incorporate uncertainty by allowing the sum of relative absolute deviations from the nominal parameter values in a constraint i to be at most Γ_i units. The value of Γ_i can vary between 0 and n_i , where n_i is the total number of uncertain parameters in the constraint i. When the value of Γ_i is n_i for all the constraints, then the polyhedral set mimics the worst-case hedge model. The robust solution is found by determining the worst-case value of the constraint while satisfying the Γ constraints. The resulting robust counterpart is said to be Γ -robust (gamma robust) and can be dualized to obtain a tractable linear formulation.

Several studies incorporate uncertainty in network flow problems. Bertsimas and Sim (2003) develop the robust minimum cost flow problem and the shortest path subject to uncertainty in link costs as a discrete optimization problem and formulate the uncertainty using the gamma robustness paradigm. Boginski et al. (2009) present a robust minimum cost flow network interdiction model considering uncertainties in interdiction probability as a linear program. The authors describe a set of failure scenarios of arc interdiction failures as in Cormican et al. (1998) and describe the conditions on the size of the failure scenario set for which the problem would be polynomially solvable. Ordóñez and Zhao (2007) study minimum cost flow problem under capacity expansion considering uncertainty in demands and travel costs. They solve the problem by employing robust optimization using ellipsoidal uncertainty sets. Minoux (2010) shows that the min-cost capacity expansion problem considering capacity uncertainty when solved using polyhedral uncertainty sets is a strongly NP-hard problem. Han et al. (2014) model a maximum flow problem for a network with uncertain arc capacities in the context of uncertainty theory. Chaerani and Roos (2007) formulate a robust maximum flow problem considering uncertainty in arc capacities using an ellipsoidal uncertainty set. Bertsimas et al. (2013) define robust and adaptive flows for a maximum flow network considering uncertainty in arc availability (similar to arc interdiction). They use polyhedral sets to restrict the maximum number of arcs that can be unavailable. They also prove that the adaptive maximum flow is always less than or equal to the maximum flow obtained after deterministic interdiction Wood (1993)]. Minoux (2009) considers a robust maximum flow problem under capacity uncertainty. When uncertainty is accounted for using polyhedral uncertainty sets, Minoux (2009) shows that the problem is strongly NP-hard. Because interdiction is considered in the current study, the worst-case outcome considered in Minoux

(2009) is different from the one considered here. Additionally, this study also considers uncertainty in resource consumption, which makes this problem at least as hard as the one considered in Minoux (2009). Therefore, the network interdiction problem to minimize the maximum flow subject to uncertainty in arc capacity and resource consumption is strongly NP-hard.

To the best of the author's knowledge, the problem considered here, the network interdiction problem with the objective of minimizing the maximum flow considering uncertainty in arc capacity and interdiction resource consumption, is *unique.* This study also provides a novel mixed-integer linear formulation to solve the problem incorporating robustness using polyhedral uncertainty sets. Three efficient heuristics based on Lagrangian Relaxation, Benders' Decomposition, and their combination are developed to improve computational times while providing quality solutions. The computational superiority of the developed heuristics is shown by competing them against a state-of-the-art solver on 31 test networks. An analysis to evaluate the performance of RNIP under various levels of uncertainty in the network parameter is carried out. A unique sensitivity analysis is conducted to identify the budgets of uncertainty which perform best using a Monte Carlo simulation scheme. Finally, an analysis to compare the robust decisions to the decisions not considering uncertainty is performed to evaluate the value of considering robustness.

3 Problem Description and Formulation

In this chapter, the robust network interdiction problem considering uncertainties in arc capacity and interdiction resource consumption (RNIP) is modeled. RNIP is formulated as a bilevel model in the attacker-defender framework. Before formulating RNIP, a brief problem description, model assumptions and the necessary nomenclature for the problem are presented. Later, the bilevel RNIP model is reformulated as a mixed-integer linear program in two ways. The chapter ends by formulating a few extensions of the current RNIP model.

3.1 Problem Description

The maximum flow network interdiction problem consists of two entities. The adversary wishes to transport as much quantity of commodity through the network as possible to maximize the commodity availability at a destination, while the interdictor wishes to interdict (or destroy) as many arcs in the network to minimize the commodity availability at the destination. Therefore, RNIP consists of two players and two decisions; the interdictor moves first and decides the arcs to interdict, whereas the adversary moves seconds, and maximizes the flow through the network.

The interdictor has complete information only about the objective of the adversary (i.e. the adversary tries to maximize the commodity availability at the destination), and the layout of the network including the source and the destination. However, the interdictor does not have complete information about the arc capacities of the network and the resource consumption required for interdiction, and therefore, these are the sources of uncertainty. It is assumed that the interdictor can estimate the arc capacities of the network using their current intelligence available about the adversary, and the demand patterns at the destination, and come up with nominal values for arc capacities and the maximum deviation from nominal values possible. Similarly, the interdictor determines the nominal resource consumption and the maximum deviation possible for the interdiction of each arc using the current intelligence available about the adversary.

The interdictor has a limited amount of resources available for interdiction. Based on the perception of the credibility of the information, the interdictor opts for using certain budgets of uncertainty for variation in arc capacity, and variation in resource consumption for interdiction. Using the above information, the interdictor wishes to determine a strategy to designate resources for interdiction in a network to minimize the maximum flow achievable by the adversary.

3.2 Model Assumptions

As RNIP is closely related to the deterministic network interdiction problem (DNIP) [Wood (1993)], many of the assumptions made in DNIP are also valid here. Assumptions involved in DNIP are detailed in Loh (1991). First, only complete interdiction is allowed, i.e., if the arc is interdicted, then it is completely destroyed and capacity drops to zero. Second, the arcs are assumed to be destroyed instantaneously once they are interdicted. The enemy can build the infrastructure back up, but it would require time much greater than the planning period considered for RNIP. Third, the arcs function independently, i.e. the destruction of an arc does not affect the performance of other arcs. Fourth, the nodes of the network are assumed to be uncapacitated. Capacity can be added to a node n by splitting it into two uncapacitated nodes n_1 and n_2 , and connecting them by a capacitated arc. Fifth, only one commodity flows through the network. Sixth, there is only a single source and a single destination in the network. The problem can be generalized for handling multiple sources and multiple destinations by connecting all sources to a super-source and all the destinations to a super-sink by infinite capacity non-interdictory arcs. It is also assumed that apart from arc interdiction, the structure of the network does not change in the game (i.e., no arcs are voluntarily abandoned, no new arcs are formed, the sources and the destinations do not change, etc.) which is implicitly assumed in most of the previous works.

The underlying non-restrictive assumptions about the maximum flow problem detailed by Ahuja et al. (1993) are also followed here. First, the network is assumed to consist only of directed arcs. Second, all the arc capacities are nonnegative integers. Third, there is no path from the source to the sink that consists only of arcs with infinite capacity. Fourth, whenever an arc from node i to node jexists, the arc from node j to node i also exists. Fifth, there are no parallel arcs in the network (i.e., no multiple arcs from node i to node j). For generalizing the above assumptions, refer to Chapter 6 in Ahuja et al. (1993).

3.3 Nomenclature

Sets

- N Set of all nodes
- A Set of all arcs
- A' Set of all arcs plus the artificial return arc, i.e. $A' = A \cup (t, s)$

 FS_i Forward star set of node *i*, i.e. set of arcs with their tails at node *i*

 RS_i Reverse star set of node *i*, i.e. set of arcs with their heads at node *i*

Indices

$$(i,j) \in A, A', FS_i, RS_i$$

 $i, j \in N$

Decision Variables

- x_{ij} Flow through the arc (i, j)
- δ_{ij} 1, if the arc (i, j) is interdicted; 0, otherwise
- γ_{ij} 1, if the arc (i, j) assumes worst-case capacity; 0, otherwise

 $\pi_{ij} \qquad 1, \text{ if the arc } (i, j) \text{ requires worst-case resource consumption for} \\ \text{interdiction; } 0, \text{ otherwise} \end{cases}$

Parameters

- *s* Source node
- t Sink node
- u_{ij} Nominal capacity of arc (i, j)

 $\hat{u}_{ij} \qquad \text{Maximum deviation from the nominal capacity of arc } (i, j), \text{ i.e. actual} \\ \text{capacity lies in the interval } [u_{ij} - \hat{u}_{ij}, u_{ij} + \hat{u}_{ij}]$

 r_{ij} Nominal amount of resource required to interdict arc (i, j)

- Maximum deviation from the nominal amount of resource required to \hat{r}_{ij} interdict arc (i, j), i.e. the actual resource required to interdict arc (i, j) lies in the interval $[r_{ij} - \hat{r}_{ij}, r_{ij} + \hat{r}_{ij}]$
- Δ Maximum amount of resource available for interdiction of arcs
- Γ The budget of uncertainty for considering robustness in capacity
- $\Pi \qquad \begin{array}{c} \text{The budget of uncertainty for considering robustness in resource} \\ \text{consumption} \end{array}$

3.4 Formulation

MODEL 1:
$$\min_{\delta} \max_{\gamma, \pi} h(\delta, \gamma)$$
 (3.4.1)

$$\sum_{(i,j)\in A'} (r_{ij} + \hat{r}_{ij}\pi_{ij})\delta_{ij} \le \Delta$$
(3.4.2)

$$\sum_{(i,j)\in A'} \pi_{ij} \le \Pi \tag{3.4.3}$$

$$\sum_{(i,j)\in A'}\gamma_{ij}\leq\Gamma\tag{3.4.4}$$

$$\delta_{ij}, \gamma_{ij}, \pi_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.5)

where,
$$h(\delta, \gamma) = \max_{x} \quad x_{ts}$$
 (3.4.6)

$$\sum_{(i,j)\in FS_i} x_{ij} - \sum_{(j,i)\in RS_i} x_{ji} = 0 \quad \forall \ i \in N$$
(3.4.7)

$$x_{ij} \le (u_{ij} + \hat{u}_{ij}\gamma_{ij})(1 - \delta_{ij}) \quad \forall \ (i,j) \in A'$$

$$(3.4.8)$$

$$x_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.9}$$

MODEL 1 represents a bilevel network interdiction problem considering uncertainties in arc capacity and resource consumption, or simply, robust network interdiction problem (RNIP). The goal of the objective function in equation 3.4.6 is to maximize the flow through the given network from source s to sink t. This total flow is then sent back to the source s from sink t through the artificial infinite capacity non-interdictory arc (t, s). Thus, flow through artificial arc (t, s)represents the total flow from source s to sink t. Constraint 3.4.7 represents the flow balance constraints across all nodes (N). Constraint 3.4.8 ensures that the flow through an arc does not exceed its realized capacity. Constraint 3.4.9 ensures flow in all the arcs are non-negative.

Objective function in equation 3.4.1 minimizes the achievable maximum flow from equation 3.4.6 by interdicting arcs while maximizing the robustness. Maximizing robustness leads to an increase in the maximum flow which is the worst-case scenario from the perspective of the local law enforcement agency, and therefore, the objective is maximized with respect to variables γ and π . Constraint 3.4.2 ensures that no more than available interdiction resource is used. Constraints 3.4.3 and 3.4.4 ensure that no more than the desired level of interdiction resource consumption robustness and capacity robustness are achieved, respectively. Constraints 3.4.5 forces the interdiction, capacity variance variables, and interdiction resource consumption variance variables to be binary. Here, the worst-case capacity is assumed to be the upper bound of the variation range, i.e. more flow is possible through the arc than expected. The worst-case resource consumption is also the upper bound of the range of variation, i.e. the resource consumption is more than expected, as it would lead to increased maximum flow. MODEL 1 represents a bi-level decision-making by two entities. Equations 3.4.1-3.4.5 represent the decision-making problem of the interdictor, i.e. maximizing

the robustness while minimizing the maximum flow, and the equations 3.4.6-3.4.9 represent the decision-making of the adversary which is to maximize the flow. In its present form, MODEL 1 can not be directly solved. MODEL 1 can be reformulated into a solvable state in two ways: first, by directly dualizing it in two stages to obtain the final form, and second, by realizing the deterministic network interdiction problem in its structure and then adding the robustness in two stages. Both ways are described in the following sections.

3.4.1 Dualizing in two stages

To solve MODEL 1, it is first dualized with respect to the variables x, and then with respect to variables γ and π . The dualization of the variables in two stages as the variable γ can only be dualized after performing the dual on the variable x. The result of dualizing with respect to x (the flow variable) yields RNIP modeled over a minimum cut problem instead of a maximum flow problem. The formulation is given as:

$$\min_{\delta,\alpha,\epsilon} \max_{\gamma,\pi} \sum_{(i,j)\in A'} (u_{ij} + \hat{u}_{ij}\gamma_{ij})(1 - \delta_{ij})\epsilon_{ij}$$
(3.4.10)

$$\sum_{(i,j)\in A'} (r_{ij} + \hat{r}_{ij}\pi_{ij})\delta_{ij} \le \Delta$$
(3.4.11)

$$\sum_{(i,j)\in A'} \pi_{ij} \le \Pi \tag{3.4.12}$$

$$\sum_{(i,j)\in A'}\gamma_{ij}\leq\Gamma\tag{3.4.13}$$

$$\alpha_i - \alpha_j + \epsilon_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.14}$$

$$\alpha_t - \alpha_s + \epsilon_{ts} \ge 1 \tag{3.4.15}$$

$$\delta_{ij}, \gamma_{ij}, \pi_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.16)

$$\alpha_i \in (-\infty, \infty) \quad \forall \ i \in N \tag{3.4.17}$$

$$\epsilon_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.18}$$

The variable α is the dual variable associated with equation 3.4.7, and the variable ϵ is the dual variable associated with equation 3.4.8. The variable α physically signifies the minimum cut of the network, dividing the network into two sets of nodes: N_s (the set of nodes containing the source node s) and N_t (the set of nodes containing the source node s) and N_t (the set of nodes containing the sink node t). Therefore, the variable α can be considered as a binary variable, with all nodes in N_s assuming the value 0 and all the nodes in N_t assuming the value 1. The variable ϵ physically signifies the forward arcs of the minimum cut of the network. As seen in the objective function 3.4.10, the product $(u_{ij} + \gamma_{ij}\hat{u}_{ij})(1 - \delta_{ij})$ is essentially the capacity of the arc. Therefore, the maximum possible value of the variable ϵ is 1. Now, ϵ can also be considered as a binary variable, with the value 1 assigned to all forward flowing arcs of the minimum cut, and 0 otherwise, without the loss of generality.

The product of two binary variables $(1 - \delta_{ij})\epsilon_{ij}$ can replaced by a new variable β_{ij} along with the constraint $\epsilon_{ij} \leq \delta_{ij} + \beta_{ij}$, as in standard linearization of binary

variables. Wood (1993) proves that the inequality constraint can be changed to $\epsilon_{ij} = \beta_{ij} + \delta_{ij}$, without the loss of optimality. Now, the variable β physically signifies the forward arcs of the min-cut that are not interdicted. After adding the above changes, the updated formulation is given as:

$$\min_{\delta,\alpha,\beta} \max_{\gamma,\pi} \sum_{(i,j)\in A'} u_{ij}\beta_{ij} + \sum_{(i,j)\in A'} \hat{u}_{ij}\beta_{ij}\gamma_{ij}$$
(3.4.19)

$$\sum_{(i,j)\in A'} r_{ij}\delta_{ij} + \sum_{(i,j)\in A'} \hat{r}_{ij}\delta_{ij}\pi_{ij} \le \Delta$$
(3.4.20)

$$\sum_{(i,j)\in A'} \pi_{ij} \le \Pi \tag{3.4.21}$$

$$\sum_{(i,j)\in A'}\gamma_{ij}\leq\Gamma\tag{3.4.22}$$

$$\alpha_i - \alpha_j + \beta_{ij} + \delta_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.23}$$

$$\alpha_t - \alpha_s + \beta_{ts} + \delta_{ts} \ge 1 \tag{3.4.24}$$

$$\delta_{ij}, \gamma_{ij}, \pi_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.25)

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{3.4.26}$$

In equation 3.4.19, the first term is the nominal capacity of a cut in the network and the second term is the increase in the cut capacity because of capacityrobustness consideration. Similarly, in equation 3.4.20, the first term and the second term refer to nominal resource consumption for interdiction and the extra resource consumption required for interdiction, respectively. If the values of parameters Π and Γ are considered to be zero, the model represents DNIP proposed in Wood (1993). Thus, the above formulation represents a DNIP integrated with arc capacity robustness and resource consumption robustness. The equation 3.4.24 can be eliminated by assigning values to all the variables involved as $\alpha_s = 0$, $\alpha_t = 1, \beta_{ts} = 0$, and $\delta_{ts} = 0$ [Bingol (2001)]. Now, the variables γ and π are dualized. The objective function for the γ variable is the second term of the equation 3.4.19, and its related constraints are equations 3.4.22 and 3.4.25. For dualizing the variable π , the objective function in π can be considered to be the maximization of the second term on the left hand side (LHS) of the equation 3.4.20, as it would lead to the increase in the overall objective function (i.e. increase in the minimum cut capacity of the network) which is the worst-case scenario in consideration. The related constraints to the variable π are equations 3.4.21 and 3.4.25. As the variables γ and π are binary, the dual is performed on its linear relaxation. The optimality can be preserved by using $|\Gamma|$ and $|\Pi|$ whenever Γ and Π are not non-negative integers. After performing dualizations with respect to variables γ and π , the final formulation is given as:

MODEL 2:
$$\min_{\alpha,\beta,\delta,\mu,\theta,\sigma,\zeta} \left(\sum_{(i,j)\in A'} u_{ij}\beta_{ij} \right) + \left(\sum_{(i,j)\in A'} \mu_{ij} \right) + \Gamma\theta \qquad (3.4.27)$$

$$\left(\sum_{(i,j)\in A'} r_{ij}\delta_{ij}\right) + \left(\sum_{(i,j)\in A'} \sigma_{ij}\right) + \Pi\zeta \le \Delta$$
(3.4.28)

$$\mu_{ij} + \theta - \hat{u}_{ij}\beta_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.29}$$

$$\sigma_{ij} + \zeta - \hat{r}_{ij}\delta_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.30}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.31}$$

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0$$
 (3.4.32)

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{3.4.33}$$

$$\delta_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A' \quad (3.4.34)$$

$$\mu_{ij}, \sigma_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.35}$$

$$\theta, \zeta \ge 0 \tag{3.4.36}$$

The dual variables θ and ζ are associated with equations 3.4.22 and 3.4.21, respectively. The dual variables μ and σ are associated with the linear relaxation of the binary variables γ and π , respectively, in equation 3.4.25. Equation 3.4.32 replaces the equation 3.4.24 from the previous formulation stage. MODEL 2 is, in fact, a modified dual of MODEL 1, and still represents RNIP which is now a mixed-integer linear program.

3.4.2 Realizing the deterministic problem and adding robustness

MODEL 1 is re-written as a bi-level problem by splitting the robustness and interdiction decisions instead of splitting it by decision-making entities. This reveals a different second-level problem, to which robustness can be added separately (detailed later in the section). This change can be mathematically represented as:

$$\max_{\gamma,\pi} \quad g(\gamma,\pi) \tag{3.4.37}$$

$$\sum_{(i,j)\in A'}\gamma_{ij}\leq\Gamma\tag{3.4.38}$$

$$\sum_{(i,j)\in A'} \pi_{ij} \le \Pi \tag{3.4.39}$$

$$\gamma_{ij}, \pi_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.40)

where,
$$g(\gamma, \pi) = \min_{\delta} \max_{x} \quad x_{ts}$$
 (3.4.41)

$$\sum_{(i,j)\in A'} \tilde{r}_{ij}\delta_{ij} \le \Delta \tag{3.4.42}$$

$$\sum_{(i,j)\in FS_i} x_{ij} - \sum_{(j,i)\in RS_i} x_{ji} = 0 \quad \forall \ i \in N$$
(3.4.43)

$$x_{ij} \le \tilde{u}_{ij}(1 - \delta_{ij}) \quad \forall \ (i, j) \in A' \tag{3.4.44}$$

$$\delta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.45)

$$x_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.46}$$

where, $\tilde{u}_{ij} = (u_{ij} + \hat{u}_{ij}\gamma_{ij})$ and $\tilde{r}_{ij} = (r_{ij} + \hat{r}_{ij}\pi_{ij})$. $g(\gamma, \pi)$, the second level of the problem, essentially represents a deterministic network interdiction problem (DNIP) [refer Wood (1993) for DNIP] subject to updated capacity \tilde{u} and updated interdiction resource consumption \tilde{r}_{ij} which are dependent on decisions made in first level (equations 3.4.37-3.4.40) of the problem. Equations 3.4.37-3.4.40 represent a robustness maximization problem in variables γ and π . As the variables γ and π are independent in the first level maximization problem, the problem can be rewritten as a bilevel problem of first determining the variable γ and then, determining the variable π , or vice versa, resulting in a trilevel problem overall. Therefore, solving RNIP directly is equivalent to adding capacity robustness and resource consumption robustness, irrespective of the order, to a DNIP. So, we first focus on solving the DNIP using nominal capacity u and nominal resource consumption r. Later, capacity robustness and resource consumption robustness are added to reformulate MODEL 1 in a format such that it can be solved. To solve the min-max objective function of DNIP, the dual is performed on variable xwhich leads to a deterministic minimum cut network interdiction problem, given as:

$$\min_{\delta,\alpha,\epsilon} \quad \sum_{(i,j)\in A'} u_{ij}(1-\delta_{ij})\epsilon_{ij} \tag{3.4.47}$$

$$\sum_{(i,j)\in A'} r_{ij}\delta_{ij} \le \Delta \tag{3.4.48}$$

$$\alpha_i - \alpha_j + \epsilon_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.49}$$

$$\alpha_t - \alpha_s + \epsilon_{ts} \ge 1 \tag{3.4.50}$$

$$\alpha_i \in (-\infty, \infty) \quad \forall \ i \in N \tag{3.4.51}$$

$$\epsilon_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.52}$$

$$\delta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A' \tag{3.4.53}$$

where, variable α represents the cut of the network by dividing it into two sets of nodes N_s (the set of nodes containing the source node s) and N_t (the set of nodes containing the sink node t). α is the dual variable associated with constraint 3.4.43. The variable ϵ represents the forward arcs of the cut and is the dual variable associated with constraint 3.4.44. It can be noticed that the objective function (constraint 3.4.47) of the min-cut network interdiction problem is non-linear. Wood (1993) modified the min-cut network interdiction problem to linearize the objective function, and proved that resulting new variable (which, we call β) and α can be assumed to be binary without the loss of optimality. This Modified Min-cut Network Interdiction Problem (MMNIP) is given as:

$$\min_{\alpha,\beta,\delta} \quad \sum_{(i,j)\in A'} u_{ij}\beta_{ij} \tag{3.4.54}$$

$$\sum_{(i,j)\in A'} r_{ij}\delta_{ij} \le \Delta \tag{3.4.55}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A$$
(3.4.56)

$$\alpha_t - \alpha_s + \delta_{ts} + \beta_{ts} \ge 1 \tag{3.4.57}$$

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{3.4.58}$$

$$\delta_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.59)

The decision variables for the MMNIP problem are:

 δ_{ij} : 1 if arc (i, j) is interdicted; 0 otherwise

 α_i : 1 if $i \in N_t$, 0 otherwise (The cut in the network is represented as (N_s, N_t))

 β_{ij} : 1 if (i,j) is a forward arc of the cut and not interdicted; 0 otherwise

The objective of the MMNIP, as shown in equation 3.4.54, is to find a minimum capacity cut of a network subject to interdiction. Bingol (2001) eliminates the equation 3.4.57 by assigning values as $\alpha_t = 1$, $\alpha_s = 0$, $\beta_{ts} = 1$, and $\delta_{ts} = 0$.

Capacity robustness is now added to the MMNIP problem, which results in the Capacity-Robust Min-Cut Network Interdiction Problem (CRMCNIP). CRMCNIP is given as:

$$\min_{\alpha,\beta,\delta} \max_{\gamma} \sum_{(i,j)\in A'} (u_{ij} + \hat{u}_{ij}\gamma_{ij})\beta_{ij}$$
(3.4.60)

$$\sum_{(i,j)\in A'}\gamma_{ij}\leq\Gamma\tag{3.4.61}$$

$$\sum_{(i,j)\in A'} r_{ij}\delta_{ij} \le \Delta \tag{3.4.62}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.63}$$

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0$$
 (3.4.64)

$$\alpha_i \in \{0,1\} \quad \forall \ i \in N \tag{3.4.65}$$

$$\gamma_{ij}, \delta_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A' \tag{3.4.66}$$

CRMCNIP is achieved by replacing the nominal capacity terms with the worst-case capacity terms and adding a constraint to adjust the level of robustness

(note that equation 3.4.61 is the same as equation 3.4.4) to MMNIP formulation. The objective function (equation 3.4.60) has a maximization with respect to the variable γ because it would lead to an increase in the maximum flow which is the worst-case scenario for RNIP. However, the above formulation can not be solved directly. CRMCNIP is updated by taking a dual with respect to the variable γ to achieve a minimization across all terms. While dualizing, the variable γ is linearized over [0,1]. Optimality of the solution is conserved by assuming Γ to be a non-negative integer, or by replacing it with $\lfloor \Gamma \rfloor$. The resulting modified dual of CRMCNIP is given as:

$$\min_{\alpha,\beta,\delta,\mu,\theta} \quad \left(\sum_{(i,j)\in A'} u_{ij}\beta_{ij}\right) + \left(\sum_{(i,j)\in A'} \mu_{ij}\right) + \Gamma\theta \tag{3.4.67}$$

$$\sum_{(i,j)\in A'} r_{ij}\delta_{ij} \le \Delta \tag{3.4.68}$$

$$\mu_{ij} + \theta - \hat{u}_{ij}\beta_{ij} \ge 0 \quad \forall \ (i,j) \in A'$$

$$(3.4.69)$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.70}$$

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0 \tag{3.4.71}$$

$$\alpha_i \in \{0,1\} \quad \forall \ i \in N \tag{3.4.72}$$

$$\delta_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.73)

$$\mu_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.74}$$

$$\theta \ge 0 \tag{3.4.75}$$

In the above formulation (equations 3.4.67 - 3.4.75), θ is the dual variable associated with equation 3.4.61 and μ_{ij} is the dual variable associated to linearization of γ_{ij} in equation 3.4.66. The first term of the objective function 3.4.67 is representative of the nominal capacity of the min-cut after interdiction, while the sum of the second and the third terms is representative of the additional capacity of the min-cut by considering capacity robustness. Adding resource consumption robustness to the modified dual of CRMCNIP results in capacity and resource consumption robust min-cut network interdiction problem, or simply, our original robust network interdiction problem (RNIP):

$$\min_{\alpha,\beta,\delta,\mu,\theta} \quad \max_{\pi} \quad \left(\sum_{(i,j)\in A'} u_{ij}\beta_{ij}\right) + \left(\sum_{(i,j)\in A'} \mu_{ij}\right) + \Gamma\theta \tag{3.4.76}$$

$$\sum_{(i,j)\in A'} (r_{ij} + \hat{r}_{ij}\pi_{ij})\delta_{ij} \le \Delta$$
(3.4.77)

$$\mu_{ij} + \theta - \hat{u}_{ij}\beta_{ij} \ge 0 \quad \forall \ (i,j) \in A'$$

$$(3.4.78)$$

$$\sum_{(i,j)\in A'} \pi_{ij} \le \Pi \tag{3.4.79}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A$$
(3.4.80)

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0 \tag{3.4.81}$$

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{3.4.82}$$

$$\delta_{ij}, \beta_{ij}, \pi_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.4.83)

$$\mu_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.84}$$

$$\theta \ge 0 \tag{3.4.85}$$

The objective function 3.4.76 is maximized with respect to π as it would lead to an increase in the maximum flow of the network, which is the worst-case scenario for RNIP. Note that equations 3.4.77 and 3.4.79 are the same as equations 3.4.2 and 3.4.3. However, the formulation can not be solved in its current form. The variable π is dualized in the same way the variable γ was dualized. Let, σ_{ij} be the dual variable associated with the linearization of π in equation 3.4.83, and ζ be the dual variable associated with the equation 3.4.79. The objective function for dualization of π can be considered to be the maximization of the second term on the LHS of the equation 3.4.77, as it would lead to an increase in the capacity of the minimum cut of the network. Non-integer values of the parameter Π are replaced by [Π] for conserve optimality while linearization. After dualizing, the resultant formulation is given as:

MODEL 2:
$$\min_{\alpha,\beta,\delta,\mu,\theta,\sigma,\zeta} \left(\sum_{(i,j)\in A'} u_{ij}\beta_{ij} \right) + \left(\sum_{(i,j)\in A'} \mu_{ij} \right) + \Gamma\theta \qquad (3.4.86)$$

$$\left(\sum_{(i,j)\in A'} r_{ij}\delta_{ij}\right) + \left(\sum_{(i,j)\in A'} \sigma_{ij}\right) + \Pi\zeta \le \Delta$$
(3.4.87)

$$\mu_{ij} + \theta - \hat{u}_{ij}\beta_{ij} \ge 0 \quad \forall \ (i,j) \in A'$$
(3.4.88)

$$\sigma_{ij} + \zeta - \hat{r}_{ij}\delta_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.89}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A \tag{3.4.90}$$

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0$$
 (3.4.91)

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{3.4.92}$$

$$\delta_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A' \quad (3.4.93)$$

$$\mu_{ij}, \sigma_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{3.4.94}$$

$$\theta, \zeta \ge 0 \tag{3.4.95}$$

In equation 3.4.87, the first term on the LHS is the nominal resource consumption for interdiction, and the sum of the second and third terms is the additional resource consumption due to the robustness considered in resource consumption. MODEL 2 is a modified dual of MODEL 1 and is written as a mixed-integer linear program.

3.5 Check for correctness of the robust formulation (MODEL 2)

To check the correctness of MODEL 2 (equations 3.4.27-3.4.36 or 3.4.86-3.4.95), consider the case when the value of the parameters Γ and Π are 0. In such a case, the values of decision variables θ and ζ can be infinitely high, making equations 3.4.29 (or 3.4.88) and 3.4.30 (or 3.4.89) redundant, and decision variables μ and σ independent from the influence of any other decision variables. Disposing the above equations also makes the decision variables θ and ζ redundant. As the objective (equation 3.4.27 or 3.4.86) is minimization, the variable μ assumes its lower bound value of 0. As the capacity of the minimum cut in the objective function (equation 3.4.27 or 3.4.86) can be reduced by increasing the number of arcs interdicted, the variables σ in equation 3.4.28 (or 3.4.87) assume their lower bound values of 0. Substituting the above changes, MODEL 2 can be written as:

MODEL 2D:

D:
$$\min_{\alpha,\beta,\delta} \left(\sum_{(i,j)\in A'} u_{ij}\beta_{ij} \right)$$
(3.5.1)

$$\left(\sum_{(i,j)\in A'} r_{ij}\delta_{ij}\right) \le \Delta \tag{3.5.2}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A$$
(3.5.3)

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0$$
 (3.5.4)

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{3.5.5}$$

$$\delta_{ij}, \beta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
 (3.5.6)

When the values of the parameters Γ and Π (budgets of uncertainty) are set to 0, it represents the case of deterministic modeling. MODEL 2D, therefore, represents a deterministic network interdiction model and is the same as one formulated in Wood (1993), i.e. the robust model reduces to a deterministic formulation when the budget of uncertainty is set to zero.

3.6 Extensions

3.6.1 Multiple resources required for interdiction

In this scenario, multiple resources are required for successful interdiction instead of a single resource. Let P be the set of different resources required for interdiction. The parameters r_{ij} , \hat{r}_{ij} , Δ , and Π are updated to r_{ijp} , \hat{r}_{ijp} , Δ_p , and Π_p to reflect resource consumption and variability, resource availability, and robustness budgets for different resources. The decision variable π_{ij} is updated to π_{ijp} to capture worst-case consumption of resource p. The updated MODEL 1 reflecting requirement of multiple resources for interdiction is given as:

$$\begin{split} \min_{\delta} \max_{\gamma, \pi} h(\delta, \gamma) \\ \sum_{(i,j) \in A'} (r_{ijp} + \hat{r}_{ijp} \pi_{ijp}) \delta_{ij} &\leq \Delta_p \quad \forall \ p \in P \\ \sum_{(i,j) \in A'} \pi_{ijp} \leq \Pi_p \quad \forall \ p \in P \\ \sum_{(i,j) \in A'} \gamma_{ij} \leq \Gamma \\ \delta_{ij}, \gamma_{ij} \in \{0, 1\} \quad \forall \ (i,j) \in A' \\ \pi_{ijp} \in \{0, 1\} \quad \forall \ (i,j) \in A', \ p \in P \end{split}$$

where, $h(\delta, \gamma) = \max_{x} \quad x_{ts}$

$$\sum_{(i,j)\in FS_i} x_{ij} - \sum_{(j,i)\in RS_i} x_{ji} = 0 \quad \forall \ i \in N$$

$$x_{ij} \le (u_{ij} + \hat{u}_{ij}\gamma_{ij})(1 - \delta_{ij}) \quad \forall \ (i, j) \in A'$$
$$x_{ij} \ge 0 \quad \forall \ (i, j) \in A'$$

The above formulation can be modified to obtain a formulation that can be solved in a commercial solver, as it was done to MODEL 1, using the methodology discussed in Subsection 3.4.1 or 3.4.2.

3.6.2 Interdiction at various levels with different economies of scale

Currently, the RNIP has only one level of interdiction, i.e. full interdiction or 100% interdiction. Discrete partial interdiction stages can be introduced to allow interdiction at various levels. This also provides an avenue to model resource consumption at those interdiction levels such that it can represent economies of scale or diseconomies of scale or both or none. Figure 3.1 shows an example illustration of how the cumulative costs of production may scale up with the total output produced to represent the above scenarios of the economy. The cumulative cost in the context of RNIP would refer to the resource requirement for interdiction at a certain level of interdiction and the total output would refer to the level of interdiction desired.

Let K be the set of interdiction levels possible. The input parameter for nominal resource consumption (r_{ij}) and variance of worst-case resource consumption



Figure 3.1: Scaling of cumulative costs with respect to total output (ES: Economies of scale, DS: Diseconomies of scale, NS: no economies of scale)

 (\hat{r}_{ij}) can now be redefined to also include interdiction levels as r_{ijk} and \hat{r}_{ijk} , respectively. A new input parameter y_k is defined which shows interdiction level as a percentage for k^{th} interdiction level, and lies in the range (0, 1]. The variable representing interdiction (δ_{ij}) can now be changed to represent interdiction at a certain level, and can be denoted as δ_{ijk} . The definition of δ_{ijk} is given as:

$$\delta_{ijk} = \begin{cases} 1 & ; \text{ if arc } (i,j) \text{ is interdicted at } k^{th} \text{ level} \\ \\ 0 & ; \text{ otherwise} \end{cases}$$

MODEL 1 can now be reformulated to observe discrete partial interdiction at the desired economy of resource consumption. This would change MODEL 1 to:

$$\begin{split} \min_{\delta} \max_{\gamma, \pi} & h(\delta, \gamma) \\ \sum_{k \in K} \sum_{(i,j) \in A'} (r_{ijk} + \hat{r}_{ijk} \pi_{ij}) \delta_{ijk} \leq \Delta \\ & \sum_{k \in K} \delta_{ijk} \leq 1 \quad \forall \ (i,j) \in A' \\ & \sum_{(i,j) \in A'} \pi_{ij} \leq \Pi \\ & \sum_{(i,j) \in A'} \gamma_{ij} \leq \Gamma \\ & \delta_{ijk} \in \{0,1\} \quad \forall \ (i,j) \in A', \ k \in K \\ & \gamma_{ij}, \pi_{ij} \in \{0,1\} \quad \forall \ (i,j) \in A' \end{split}$$

where, $h(\delta, \gamma) = \max_{x} \quad x_{ts}$

$$\sum_{(i,j)\in FS_i} x_{ij} - \sum_{(j,i)\in RS_i} x_{ji} = 0 \quad \forall \ i \in N$$
$$x_{ij} \le (u_{ij} + \hat{u}_{ij}\gamma_{ij})(1 - \sum_{k\in K} y_k\delta_{ijk}) \quad \forall \ (i,j) \in A'$$
$$x_{ij} \ge 0 \quad \forall \ (i,j) \in A'$$

The above model can not be solved directly, but can be reformulated using the procedure presented in Subsection 3.4.1 to obtain a mixed-integer linear program which can be solved.

3.7 Summary

In this chapter, RNIP is modeled in the context of illicit drugs as a bilevel attackerdefender framework. The problem was then reformulated as a mixed-integer linear program in two ways arriving at the same final formulation. The first way was straight-forward dualization which revealed that RNIP is an integrated robustness decision and DNIP framework. In the second way, it was also established that "adding" robustness incrementally to the DNIP would result in the same model as the integrated model obtained using the first way of derivation. The resulting RNIP is a mixed-integer linear program which can be solved using commercially available MIP solvers like Gurobi, CPLEX, etc. A few extensions of the current RNIP model were also discussed.

Solving RNIP directly using the MIP solver can be computationally expensive and hence, the next chapter discusses various solution heuristics to solve RNIP in a time-efficient manner while preserving the solution quality. Solution heuristics based on Lagrangian Relaxation and Benders' Decomposition are proposed. Valid upper bounds of the dual variables of the robustness decision variables γ and π are also derived.

4 Solution Methodology

The maximum flow network interdiction problem considering uncertainties in arc capacity and resource consumption is abbreviated as robust network interdiction problem (RNIP), and is described by MODEL 2 formulation (equations 3.4.86-3.4.95). This chapter describes three heuristics to solve RNIP: the first heuristic is based on Lagrangian Relaxation, the second is based on Benders' Decomposition initialized using nominal capacity and resource consumption constraints, and the third heuristic is based on Benders' Decomposition initialized using capacity and resource consumption constraints which are found using Lagrangian Relaxation. Before describing the heuristics, valid upper bounds for the unbounded variables of RNIP are derived, and the procedure used to solve robust knapsack problems considering item weight uncertainty is reviewed.

4.1 Valid Upper Bounds

Though the variables μ , σ , θ and ζ are right-unbounded after dualizing the constraints involving γ and π variables (refer MODEL 2 in Section 3.4.1 or 3.4.2), their natural upper bounds exist and are derived later in this section. The upper bounds are given as:

$$\mu_{ij} \le \hat{u}_{ij} \ \forall \ (i,j) \in A' \tag{4.1.1}$$

$$\theta \le \max\{\hat{u}_{ij} \mid (i,j) \in A\} = \theta_U \tag{4.1.2}$$

$$\sigma_{ij} \le \hat{r}_{ij} \ \forall \ (i,j) \in A' \tag{4.1.3}$$

$$\zeta \le \max\{\hat{r}_{ij} \mid (i,j) \in A^Z\} = \zeta_U \tag{4.1.4}$$

where, A^Z is the set of all arcs that can be interdicted while considering resource consumption robustness, i.e. $A^Z = \{(i,j) | (r_{ij} + \hat{r}_{ij}) \leq \Delta \forall (i,j) \in A'\}$ (note, $A^Z \subseteq A$).

Constraints 4.1.1-4.1.4 can be added to MODEL 2 to obtain stronger cuts on variables μ , θ , σ , and ζ . It is shown later that while applying Lagrangian Relaxation, these constraints also help in avoiding unboundedness in the model.

The intuition behind obtaining the bounds for the μ and θ variables is found in the objective function of MODEL 2 (equation 3.4.27 or 3.4.86) and the definition of Γ . The sum of the second and the third term of the objective function is equal to the increase in capacity of the network because of adding capacity robustness. The number Γ signifies the maximum number of arcs that can achieve their worst-case capacities. Similarly, the bounds on the variables for the σ and ζ variables are found using the interdiction budget constraint in MODEL 2 (equation 3.4.28 or 3.4.87) and the definition of Π . The sum of the second and the third terms on the LHS of the interdiction budget constraint signifies the additional resource consumption because of the consideration of interdiction resource consumption robustness. The variable Π denotes the maximum number of arcs that can achieve worst-case resource consumption when they are interdicted.

Actual Upper bounds on μ and σ :

The linearization of variable γ in equation 3.4.25 (or 3.4.66) is given as:

$$\gamma_{ij} \le 1 \quad \forall \ (i,j) \in A' \tag{4.1.5}$$

 μ is the dual variable corresponding to constraint 4.1.5. The upper bound on μ can be obtained by studying the effect of constraint 4.1.5 on equation 3.4.19 (or 3.4.60), which results in:

$$\mu_{ij} \le \hat{u}_{ij} \ \forall \ (i,j) \in A'$$

Similarly, the linearization of the variable π in equation 3.4.25 (or 3.4.83) is given as:

$$\pi_{ij} \le 1 \quad \forall \ (i,j) \in A' \tag{4.1.6}$$

 σ is the dual variable variable corresponding to constraint 4.1.6. The upper bound on σ is obtained by studying the effect of equation 4.1.6 on equation 3.4.20 (or 3.4.77), which results in:

$$\pi_{ij} \le \hat{r}_{ij} \ \forall \ (i,j) \in A^{i}$$

An Upper Bound on θ :

All the forward-flowing arcs of a cut (N_s, N_t) in the network can have their capacities increased. Given any network, we can be sure that the artificial return arc can never be a forward-flowing arc of a cut. Let us consider the capacity variances of all the network arcs except the artificial return arc (i.e. the arc set A), in a non-increasing order:

$$\hat{u}_1 \ge \hat{u}_2 \ge \hat{u}_3 \ge \ldots \ge \hat{u}_{\Gamma} \ge \ldots \ge \hat{u}_{|A|-1} \ge \hat{u}_{|A|}$$

The maximum increase possible in the capacity of the network is the sum of the first Γ values of series mentioned, i.e. $\sum_{k=1}^{\Gamma} \hat{u}_k$. This can be represented mathematically as:

$$\left(\sum_{(i,j)\in A'}\mu_{ij}\right) + \Gamma\theta \le \sum_{k=1}^{\Gamma}\hat{u}_k \tag{4.1.7}$$

To calculate an upper bound on θ , we utilize the following inequality:

$$\Gamma \hat{u}_1 \ge \sum_{k=1}^{\Gamma} \hat{u}_k \tag{4.1.8}$$

Using equations 4.1.7 and 4.1.8, and assuming all $\mu = 0$, an upper bound on θ is found to be:

$$\theta \le \hat{u}_1 = \max\{\hat{u}_{ij} \mid (i,j) \in A\}$$

An Upper Bound on ζ :

The resource consumption robustness will be considered for an arc only if it can be interdicted, making it a prerequisite for resource consumption robustness decision. Let us consider the resource consumption variances of all the network arcs that can be interdicted (i.e. the arc set A^Z) in a non-increasing order, i.e.

$$\hat{r}_1 \ge \hat{r}_2 \ge \hat{r}_3 \ge \ldots \ge \hat{r}_{\Pi} \ge \ldots \ge \hat{r}_{|A^Z|-1} \ge \hat{r}_{|A^Z|}$$

The maximum increase possible in the capacity of the network is the sum of the first Π values of series mentioned, i.e. $\sum_{k=1}^{\Pi} \hat{r}_k$. This can be represented mathematically as:

$$\left(\sum_{(i,j)\in A'}\sigma_{ij}\right) + \Pi\zeta \le \sum_{k=1}^{\Gamma}\hat{r}_k \tag{4.1.9}$$
To calculate an upper bound on θ , we utilize the following inequality:

$$\Pi \hat{r}_1 \ge \sum_{k=1}^{\Pi} \hat{r}_k \tag{4.1.10}$$

Using equations 4.1.9 and 4.1.10, and assuming all $\sigma=0$, an upper bound on ζ is found to be:

$$\zeta \le \hat{r}_1 = \max\{\hat{r}_{ij} \mid (i,j) \in A^Z\}$$

4.2 Robust Knapsack Problem

Knapsack problem with item weight uncertainty was first solved using Γ -robustness in Bertsimas and Sim (2003). Bertsimas and Sim (2003) named the problem the Robust Knapsack Problem (RKP) and solved it directly using CPLEX. The formulation of the RKP with integer values of the budget of uncertainty Γ is given as:

$$\max_{x} \sum_{i \in N} c_{i} x_{i}$$
$$\sum_{i \in N} w_{i} x_{i} + \max_{\{S \mid S \subseteq N, |S| = \Gamma\}} \left(\sum_{j \in S} \hat{w}_{j} x_{j} \right) \leq b$$
$$x_{i} \in \{0, 1\} \quad \forall \ i \in N$$

where, N denotes the set of all items, c_i denotes item values, the item weights

can vary between $[w_i - \hat{w}_i, w_i + \hat{w}_i]$ (w_i being the nominal item weight and \hat{w}_i being the item weight variance), and b denotes the knapsack capacity.

Monaci et al. (2013) proposed a dynamic programming approach to solve the RKP and compared it with three other approaches: MILP formulation by Bertsimas and Sim (2004), Branch and Cut enumeration, and procedure proposed by Lee et al. (2012). Their approach performed better than all the other three approaches when the number of items and the knapsack capacities were high with lower values of uncertainty budget Γ . The approach proposed by Lee et al. (2012) performed slightly better than the dynamic programming approach when the number of items and the knapsack capacity was less or when the higher budget of uncertainty Γ was used. The MILP formulation and Branch and Cut enumeration performed significantly worse than both of the other approaches. In the current study, the number of items in the knapsack is low with a small budget of uncertainty, and therefore the approach proposed by Lee et al. (2012) is chosen for implementation.

Lee et al. (2012) proved that the RKP can be solved by solving $(|N| - \Gamma + 1)$ instances of the ordinary 0-1 knapsack problem, where |N| is the total number of items and Γ is the budget of robustness. We just focus on the solving methodology here, and not the proof. The feasible region (S) for the 0-1 RKP is given as:

$$S = \left\{ x \in \{0,1\}^n \ \left| \ \sum_{i \in N} w_i x_i + \max_{T \subseteq N, |T| = \Gamma} \sum_{j \in T} \hat{w}_j x_j \le b \right\} \right.$$
$$= \left\{ x \in \{0,1\}^n \ \left| \ \sum_{i \in N} w_i x_i + \sum_{j \in U} \hat{w}_j x_j \le b \ , \ \forall \ U \subseteq N \text{ with } |U| = \Gamma \right\} \right.$$

where, $N = \{1, 2, 3, ..., n\}$, w and \hat{w} are non-negative integers, and items arranged such that

 $\hat{w}_1 \ge \hat{w}_2 \ge \ldots \ge \hat{w}_n$. An additional item n+1 is added with $\hat{w}_{n+1} = 0$, $w_{n+1} = 0, c_{n+1} = 0$, and the set of items is updated as: $N^+ = N \cup \{n+1\}$. Set

$$L = \{\Gamma, \Gamma + 1, \Gamma + 2, \dots, n, n + 1\}$$
$$S_l = \left\{ x \in \{0, 1\}^n \ \left| \ \sum_{i \in N} w_i x_i + \sum_{j \in N_l} (\hat{w}_j - \hat{w}_l) x_j \le b - \Gamma \hat{w}_l \ , \ N_l = \{j \in N^+ \mid j \le l\} \right\}$$

The RKP, given as:

$$Z^* = \max\left\{\sum_{i \in N} c_i x_i \ \middle| \ x \in S\right\}$$

can be solved by solving $(|N|-\Gamma+1)$ ordinary 0-1 knapsack problems

$$Z_l^* = \max\left\{\sum_{i \in N} c_i x_i \ \middle| \ x \in S_l\right\}, \ \forall \ l \in L$$
$$Z^* = \max\left\{Z_l^* \ \forall \ l \in L\right\}$$

The ordinary 0-1 knapsack problem can be solved using dynamic programming or by using a MIP solver like Gurobi, CPLEX, etc. Some preliminary analysis showed that Gurobi was found to be more computationally efficient than dynamic programming and therefore, Gurobi was chosen to solve the ordinary 0-1 knapsack problem for all runs.

4.3 Lagrangian Relaxation

Bingol (2001) suggested a Lagrangian Relaxation formulation for the MMNIP (the modified dual of DNIP) to find problematic values of interdiction resource Δ . This served as a motivation to try Lagrangian Relaxation to solve the RNIP. First, constraint 3.4.87 (or 3.4.28) is relaxed using Lagrangian parameter $\lambda' \geq 0$, constraint 3.4.88 (or 3.4.29) is relaxed using Lagrangian parameter $\lambda''_{ij} \geq 0$, and constraint 3.4.89 (or 3.4.30) is relaxed using Lagrangian parameter $\lambda''_{ij} \geq 0$ in the final RNIP formulation (MODEL 2). The upper bounds on the variables μ , θ , σ , and ζ found in section 4.1 are also added. This results in:

$$\min_{\alpha,\beta,\delta,\mu,\theta,\sigma,\zeta} \quad \left(\sum_{(i,j)\in A'} (u_{ij} + \lambda_{ij}''\hat{u}_{ij})\beta_{ij} \right) + \left(\sum_{(i,j)\in A'} (\lambda'r_{ij} + \lambda_{ij}'''\hat{r}_{ij})\delta_{ij} \right) - \lambda'\Delta + \left(\sum_{(i,j)\in A'} (1 - \lambda_{ij}'')\mu_{ij} \right) + \left(\Gamma - \sum_{(i,j)\in A'} \lambda_{ij}'' \right) \theta + \left(\sum_{(i,j)\in A'} (\lambda' - \lambda_{ij}'')\sigma_{ij} \right) + \left(\lambda'\Pi - \sum_{(i,j)\in A'} \lambda_{ij}''' \right) \zeta$$

$$\begin{aligned} \alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} &\geq 0 \quad \forall \ (i,j) \in A \\ \alpha_s &= 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0 \\ \alpha_i \in \{0,1\} \quad \forall \ i \in N \\ \delta_{ij}, \beta_{ij} \in \{0,1\} \quad \forall \ (i,j) \in A' \\ \mu_{ij} \in [0, \hat{u}_{ij}] \quad \forall \ (i,j) \in A' \\ \theta \in [0, \theta_U] \\ \sigma_{ij} \in [0, \hat{r}_{ij}] \quad \forall \ (i,j) \in A' \\ \zeta \in [0, \zeta_U] \end{aligned}$$

Because Lagrangian Relaxation is a relaxation, i.e., it increases the feasible region for the problem, the new optimum will be lower than the optimum of the original problem. The above formulation is modified to consider λ' , λ''_{ij} , and λ'''_{ij} as variables and maximizing with respect to them to achieve the maximum possible value of the relaxed problem, i.e. the best lower bound possible. This problem is essentially the Lagrangian dual of the relaxed RNIP.

$$\max_{\lambda',\lambda''} \min_{\alpha,\beta,\delta,\mu,\theta,\sigma,\zeta} \left(\sum_{(i,j)\in A'} (\bar{u}_{ij} + \lambda''_{ij}\hat{u}_{ij})\beta_{ij} \right) + \left(\sum_{(i,j)\in A'} (\lambda'r_{ij} + \lambda''_{ij}\hat{r}_{ij})\delta_{ij} \right) - \lambda'\Delta + \left(\sum_{(i,j)\in A'} (1 - \lambda''_{ij})\mu_{ij} \right) + \left(\Gamma - \sum_{(i,j)\in A'} \lambda''_{ij} \right) \theta + \left(\sum_{(i,j)\in A'} (\lambda' - \lambda''_{ij})\sigma_{ij} \right) + \left(\Pi\lambda' - \sum_{(i,j)\in A'} \lambda''_{ij} \right) \zeta$$

$$\begin{aligned} \alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} &\geq 0 \quad \forall \ (i,j) \in A \\ \alpha_s &= 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0 \\ \alpha_i \in \{0,1\} \quad \forall \ i \in N \\ \delta_{ij}, \beta_{ij} \in \{0,1\} \quad \forall \ (i,j) \in A' \\ \mu_{ij} \in [0, \hat{u}_{ij}] \quad \forall \ (i,j) \in A' \\ \theta \in [0, \theta_U] \\ \sigma_{ij} \in [0, \hat{r}_{ij}] \quad \forall \ (i,j) \in A' \\ \zeta \in [0, \zeta_U] \end{aligned}$$

The above formulation can be decomposed into 5 sub-problems: first containing α, β , and δ terms, and the other four are one-variable sub-problems in the variables μ, θ, σ , and ζ ; if λ', λ'' , and λ''' are considered constants. A gradientbased optimization method can then be used to solve for the Lagrangian parameters in the Lagrangian Dual as suggested by Fisher (1985). The sub-problems are given as:

Sub-problem 1:

$$\min_{\alpha,\beta,\delta} \left(\sum_{(i,j)\in A'} (u_{ij} + \lambda_{ij}''\hat{u}_{ij})\beta_{ij} + (\lambda'r_{ij} + \lambda_{ij}'''\hat{r}_{ij})\delta_{ij} \right) - \lambda'\Delta$$
$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A$$
$$\alpha_t - \alpha_s + \delta_{ts} + \beta_{ts} \ge 1$$

$$\begin{aligned} \alpha_s &= 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0 \\ \alpha_i &\in \{0, 1\} \quad \forall \ i \in N \\ \delta_{ij}, \beta_{ij} &\in \{0, 1\} \quad \forall \ (i, j) \in A' \end{aligned}$$

The dual of the sub-problem 1 results in a maximum flow problem with updated capacities, as in Bingol (2001).

$$SP_{1} = \max_{x} \quad x_{ts} - \lambda' \Delta$$

$$\sum_{(i,j)\in FS_{i}} x_{ij} - \sum_{(j,i)\in RS_{i}} x_{ji} = 0 \quad \forall \ i \in N$$

$$x_{ij} \leq \min(u_{ij} + \lambda''_{ij}\hat{u}_{ij}, \ \lambda'r_{ij} + \lambda''_{ij}\hat{r}_{ij}) \quad \forall \ (i,j) \in A'$$

$$x_{ij} \geq 0 \quad \forall \ (i,j) \in A'$$

Sub-problem 2:

$$SP_2 = \min_{\mu} \left(\sum_{(i,j)\in A'} (1 - \lambda_{ij}'') \mu_{ij} \right)$$
$$\mu_{ij} \in [0, \hat{u}_{ij}] \quad \forall \ (i,j) \in A'$$

The solution to the above problem is trivial.

$$\mu_{ij} = \begin{cases} \hat{u}_{ij} & ; \text{ if } \lambda_{ij}'' \ge 1 \\ 0 & ; \text{ otherwise} \end{cases} \quad \forall \ (i,j) \in A'$$

Sub-problem 3:

$$SP_{3} = \min_{\theta} \left(\Gamma - \sum_{(i,j) \in A'} \lambda_{ij}'' \right) \theta$$
$$\theta \in [0, \theta_{U}]$$

The solution to the above problem is trivial.

$$\theta = \begin{cases} \theta_U & \text{; if } \sum_{(i,j) \in A'} \lambda_{ij}'' \ge \Gamma \\ 0 & \text{; otherwise} \end{cases}$$

Sub-problem 4:

$$SP_4 = \min_{\sigma} \left(\sum_{(i,j)\in A'} (\lambda' - \lambda'''_{ij}) \sigma_{ij} \right)$$
$$\sigma_{ij} \in [0, \hat{r}_{ij}] \quad \forall \ (i,j) \in A'$$

The solution to the above problem is trivial.

$$\sigma_{ij} = \begin{cases} \hat{r}_{ij} & ; \text{ if } \lambda_{ij}^{\prime\prime\prime} \ge \lambda' \\ 0 & ; \text{ otherwise} \end{cases} \quad \forall \ (i,j) \in A'$$

Sub-problem 5:

$$SP_5 = \min_{\zeta} \left(\Pi \lambda' - \sum_{(i,j) \in A'} \lambda'''_{ij} \right) \zeta$$
$$\zeta \in [0, \zeta_U]$$

The solution to the above problem is trivial.

$$\zeta = \begin{cases} \zeta_U & \text{; if } \sum_{(i,j) \in A'} \lambda_{ij}^{\prime\prime\prime} \ge \Pi \lambda' \\ 0 & \text{; otherwise} \end{cases}$$

Note that without the upper bounds found in Section 4.1, the sub-problems 2-5 could result in unbounded solutions.

4.3.1 Solution Procedure

- 1. Initialize the Lagrangian parameters; $\lambda' \leftarrow \max(u/r \mid u < M; r < M)$,
 - $\lambda_{ij}'' \leftarrow 0 \forall (i,j) \in A'$, and $\lambda_{ij}''' \leftarrow 0 \forall (i,j) \in A'$. Set the tolerance value, tol, (use 10^{-4} as default value), maximum iterations limit (use 20 iterations as

default value) for the problem, and maximum computational time limit (use 21600 sec as default value). Set $Z_{ub}^{old} = \infty$ and $Z_{lb}^{old} = 0$.

- Solve for SP₁, which is a maximum flow problem to obtain optimal value of x, which can be denoted by x̂. Using the dual of the problem, the optimal cut is obtained, which is denoted by α̂. Solve for SP₂, SP₃, SP₄, and SP₅, which are unconstrained optimization problems with bounds on variables μ, θ, σ, and ζ, respectively. The optimal values of these variables can be denoted by μ̂, θ̂, σ̂, and ζ̂.
- 3. The values of $\hat{\delta}$ are calculated as:

$$\hat{\delta}_{ij} = \begin{cases} 1 \quad ; \text{ if } \hat{\alpha}_j - \hat{\alpha}_i = 1 \& (u_{ij} + \lambda_{ij}'' \hat{u}_{ij}) > (\lambda' r_{ij} + \lambda_{ij}''' \hat{r}_{ij}) \\ 0 \quad ; \text{ otherwise} \end{cases} \quad \forall \ (i,j) \in A'$$

4. The values of $\hat{\beta}$ are calculated as:

$$\hat{\beta}_{ij} = \begin{cases} 1 & \text{; if } \hat{\alpha}_j - \hat{\alpha}_i = 1 \& \hat{\delta}_{ij} = 0 \\ 0 & \text{; otherwise} \end{cases} \quad \forall (i,j) \in A'$$

5. Calculate the lower bound Z_{lb} as:

$$Z_{lb} = SP_1 + SP_2 + SP_3 + SP_4 + SP_5$$

If $Z_{lb} \geq Z_{lb}^{old}$, update $Z_{lb}^{old} = Z_{lb}$ and store the corresponding optimal variable

values $\hat{\alpha}$, $\hat{\beta}$, $\hat{\delta}$, $\hat{\mu}$, $\hat{\theta}$, $\hat{\sigma}$, and $\hat{\zeta}$. Otherwise, update the current Z_{lb} with Z_{lb}^{old} and the new variable optimum values with ones corresponding to Z_{lb}^{old} .

6. The next goal is to utilize the Lagrangian solution to find a feasible solution (denoted by variables α, β, δ, μ, and θ) which will help determine the upper bound, Z_{ub}. The fact that the robustness and the interdiction decisions are based on the minimum cut of the network is exploited to obtain the upper bound solution. The first step is to determine the minimum cut (N_s, N_t) and minimum cut forward flowing arc set A_C, utilizing the dual values of sub-problem 1. Set:

$$\alpha_{i} = \hat{\alpha}_{i} = \begin{cases} 1 \quad ; \text{ if } i \in N_{t} \\\\ 0 \quad ; \text{ if } i \in N_{s} \end{cases}$$
$$A_{C} = \left\{ (i, j) \mid \alpha_{j} - \alpha_{i} = 1 \forall (i, j) \in A' \right\}$$

- 7. All the arcs in the set A_C that can be interdicted are considered to be a part of the interdiction set A_I , i.e. $A_I = \{(i, j) | (r_{ij} + \hat{r}_{ij}) \leq \Delta \forall (i, j) \in A_C \}.$
 - (a) Initialize: $\delta_{ij} = 0 \ \forall \ (i,j) \in A'$.
 - (b) Arrange the resource consumption variances for all arcs in A_I in a nonincreasing order, i.e. $\hat{r}_1 \ge \hat{r}_2 \ge \ldots \ge \hat{r}_{|A_I|}$ (henceforth, denoted as \hat{r}_k).

(c) If
$$\sum_{(i,j)\in A_I} r_{ij} + \sum_{k=1}^{\Pi} \hat{r}_k \leq \Delta$$
:

Set
$$\delta_{ij} = 1 \ \forall \ (i,j) \in A_I$$
.

else:

- i. Solve the robust knapsack problem (RKP) discussed in Section 4.2, with the set $N = A_I$, and the parameters $c = (u + \lambda''\hat{u}), b = \Delta$, $w = r, \hat{w} = \hat{r}$, and $\Gamma = \Pi$.
- 8. Set:

$$\beta_{ij} = \begin{cases} 1 & ; \text{ if } (i,j) \text{ is a forward arc of the cut } (N_s, N_t) \text{ and } \delta_{ij} = 0 \\ 0 & ; \text{ otherwise} \end{cases}$$

9. Let, $\kappa = \min \{ \Gamma, \sum_{(i,j) \in A_C} \beta_{ij} \}$. Set θ as the κ^{th} largest $\hat{u}_{ij}\beta_{ij}$ value.

10. Set
$$\mu_{ij} = \max(0, \ \hat{u}_{ij}\beta_{ij} - \theta) \quad \forall \ (i,j) \in A'$$

11. Calculate the upper bound, Z_{ub} as:

$$Z_{ub} = \left(\sum_{(i,j)\in A'} u_{ij}\beta_{ij}\right) + \left(\sum_{(i,j)\in A'} \mu_{ij}\right) + \Gamma\theta$$

If $Z_{ub} \leq Z_{ub}^{old}$, update $Z_{ub}^{old} = Z_{ub}$ and store the corresponding optimal variable values α , β , δ , μ and θ . Otherwise, update the current Z_{ub} with Z_{ub}^{old} and the new variable optimum values with ones corresponding to Z_{ub}^{old} . 12. Calculate the MIPGap as:

$$MIPGap = \frac{Z_{ub} - Z_{lb}}{Z_{lb}}$$

- 13. If $MIPGap \leq tol$, then Z_{ub} is the optimal objective value and the values of $\alpha, \beta, \delta, \mu$, and θ constitute the optimal solution. If the maximum number of iterations is achieved or the time limit is exceeded, then it is concluded that no optimal solution is found and the current upper bound solution is reported. Continue to Step 16.
- 14. Update the Lagrangian parameters λ' , λ'' , and λ''' using the updating procedure in Subsection 4.3.2.
- 15. Repeat steps 2-14 until one of the conditions in Step 13 is satisfied.
- 16. The feasible values of the variables ζ and σ are given as:
 - (a) Let, $\phi = \min \{ \prod, \sum_{(i,j) \in A_I} \delta_{ij} \}$. Set ζ value as ϕ^{th} largest $\hat{r}_{ij} \delta_{ij}$ value.
 - (b) Set $\sigma_{ij} = \max(0, \ \hat{r}_{ij}\delta_{ij} \zeta) \quad \forall \ (i, j) \in A'.$

4.3.2 Lagrangian Parameters' Updating Procedure

The procedure for updating the Lagrangian parameter is the same as in Fisher (1985):

$$\lambda'|_{n+1} = \max\left\{0, \lambda'|_n + f_n^{\lambda'} \cdot \left(-\Delta + \left(\sum_{(i,j)\in A'} r_{ij}\hat{\delta}_{ij}\right) + \left(\sum_{(i,j)\in A'} \hat{\sigma}_{ij}\right) + \Gamma\hat{\theta}\right)\right|_n\right\}$$
$$\lambda''_{ij}|_{n+1} = \max\left\{0, \lambda''_{ij}|_n + f_n^{\lambda''} \cdot \left(\hat{u}_{ij}\hat{\beta}_{ij} - \hat{\mu}_{ij} - \hat{\theta}\right)\Big|_n\right\} \quad \forall \ (i,j) \in A'$$
$$\lambda'''_{ij}|_{n+1} = \max\left\{0, \lambda'''_{ij}|_n + f_n^{\lambda'''} \cdot \left(\hat{r}_{ij}\hat{\delta}_{ij} - \hat{\sigma}_{ij} - \hat{\zeta}\right)\Big|_n\right\} \quad \forall \ (i,j) \in A'$$

where, $f_n^{\lambda'}$, $f_n^{\lambda''}$ and $f_n^{\lambda'''}$ are given as:

$$f_n^{\lambda'} = \frac{g_n^{\lambda'}(Z_{ub} - Z_{lb})}{\left(-\Delta + \left(\sum_{(i,j)\in A'} r_{ij}\hat{\delta}_{ij}\right) + \left(\sum_{(i,j)\in A'} \hat{\sigma}_{ij}\right) + \Gamma\hat{\theta}\right)^2}$$
$$f_n^{\lambda''} = \frac{g_n^{\lambda''}(Z_{ub} - Z_{lb})}{\sum\limits_{(i,j)\in A'} \left(\hat{u}_{ij}\hat{\beta}_{ij} - \hat{\mu}_{ij} - \hat{\theta}\right)^2}$$
$$f_n^{\lambda'''} = \frac{g_n^{\lambda'''}(Z_{ub} - Z_{lb})}{\sum\limits_{(i,j)\in A'} \left(\hat{r}_{ij}\hat{\delta}_{ij} - \hat{\sigma}_{ij} - \hat{\zeta}\right)^2}$$

In the above equations, 'n' represents the iteration number. $g_n^{\lambda'}$, $g_n^{\lambda''}$, and $g_n^{\lambda'''}$ are initialized at 0.1. They are halved every time the Z_{lb} fails to increase in two iterations.

4.4 Benders' Decomposition-based Heuristic

Benders' Decomposition is a partitioning algorithm developed by Benders (1962) to tackle "complicating" variables by temporarily fixing them, which yields a problem significantly easy to solve. Rahmaniani et al. (2017) conducted a comprehensive literature review on the Benders' Decomposition algorithm detailing various methods for solution procedure, solution generation, decomposition strategy, and cut generation. For RNIP, the Benders' Decomposition Master Problem and the Sub-Problem are formulated by fixing the values of only β variables, as it results in a sub-problem that generates only optimality cuts and no feasibility cuts. This is desired as it allows for the evaluation of upper bound and lower bound of RNIP after each iteration. Benders' Decomposition Sub-Problem for RNIP is given as:

Sub-Problem:
$$\min_{\mu,\theta} \left(\sum_{(i,j)\in A'} \mu_{ij} \right) + \Gamma \theta$$
 (4.4.1)

$$\mu_{ij} + \theta \ge \hat{u}_{ij}\bar{\beta}_{ij} \quad \forall \ (i,j) \in A' \tag{4.4.2}$$

$$\mu_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{4.4.3}$$

$$\theta \ge 0 \tag{4.4.4}$$

The dual of the sub-problem is written adopting γ_{ij} as a dual variable for constraint 4.4.2. This is done because the variables μ and θ were generated when the constraints involving the variable γ were dualized while formulating RNIP (refer section 3.4.1 or 3.4.2). Therefore, the meaning of the variable γ here is the same here as the one noted in the nomenclature of variables in section 3.3. The dual of the Benders' Sub-Problem is given as:

Sub-Problem-Dual:
$$\max_{\gamma} \left(\sum_{(i,j)\in A'} \hat{u}_{ij} \bar{\beta}_{ij} \gamma_{ij} \right)$$
$$\sum_{(i,j)\in A'} \gamma_{ij} \leq \Gamma$$
$$\gamma_{ij} \leq 1 \quad \forall \ (i,j) \in A'$$
$$\gamma_{ij} \geq 0 \quad \forall \ (i,j) \in A'$$

Because Γ is assumed to be a non-negative integer, the sub-problem dual is essentially a 0-1 knapsack problem with all weight values of γ variables as 1. This can be solved by just setting $\gamma_{ij} = 1$ for the Γ highest $\hat{u}_{ij}\bar{\beta}_{ij}$ values, and $\gamma_{ij} = 0$ for all other arcs of the network. The problem is always feasible and yields optimum. Therefore, it always results in an optimality cut, which is added to the Master Problem. This optimality cut is given as:

$$\sum_{(i,j)\in A'} (u_{ij} + \gamma^o_{ij}\hat{u}_{ij})\beta_{ij} - z \le 0$$

where, γ_{ij}^{o} represents the γ_{ij} values for the optimality cut o. The Benders' Master Problem along with the optimality cuts is given as:

Master Problem:
$$\min_{z,\alpha,\beta,\delta} z$$
 (4.4.5)

$$\left(\sum_{(i,j)\in A'} (u_{ij} + \gamma^o_{ij}\hat{u}_{ij})\beta_{ij}\right) - z \le 0 \quad \forall \ o \in O$$

$$(4.4.6)$$

$$\left(\sum_{(i,j)\in A'} r_{ij}\delta_{ij}\right) + \left(\sum_{(i,j)\in A'} \sigma_{ij}\right) + \Pi\zeta \le \Delta$$
(4.4.7)

$$\sigma_{ij} + \zeta - \hat{r}_{ij}\delta_{ij} \ge 0 \quad \forall \ (i,j) \in A' \tag{4.4.8}$$

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A$$

$$(4.4.9)$$

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0$$
 (4.4.10)

$$z \in (-\infty, \infty) \tag{4.4.11}$$

$$\alpha_i \in \{0, 1\} \quad \forall \ i \in N \tag{4.4.12}$$

$$\beta_{ij}, \delta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
(4.4.13)

O represents the set of all optimality cuts generated. A heuristic is designed to solve the Benders' Decomposition Master Problem which is presented in the following section.

4.4.1 A Simultaneous Penalty Heuristic for solving the Master Problem

Though the Master Problem has a reduced amount of constraints than the original problem, the inherent complexity of the problem is still the same. Therefore, the computational time required to solve the Master Problem exactly will be of the same order as solving the original problem exactly, i.e., it would be of the order of the MIP solver's computational times. Also, as Benders' Decomposition is an iterative procedure with an increasing amount of constraints with each iteration, the computational times would just pile up with each passing iteration, making it an unattractive option. Therefore, a heuristic is designed for quick computation of the Master Problem of Benders' Decomposition. It should be noted that the heuristic solution results in an upper bound for the Master Problem. The equivalent formulation of the Master Problem of Benders' Decomposition as noted in equations 4.4.5-4.4.13, is given as:

Master Problem:
$$\min_{z,\alpha,\beta,\delta} z$$
 (4.4.14)

$$\left(\sum_{(i,j)\in A'} (u_{ij} + \hat{u}_{ij}\gamma^o_{ij})\beta_{ij}\right) - z \le 0 \quad \forall \ o \in O$$

$$(4.4.15)$$

$$\left(\sum_{(i,j)\in A'} r_{ij}\delta_{ij}\right) + \left(\sum_{(i,j)\in S} \hat{r}_{ij}\delta_{ij}\right) \le \Delta \quad \forall \ \{S \mid S \subseteq A', |S| = \Pi\}$$
(4.4.16)

$$\alpha_i - \alpha_j + \delta_{ij} + \beta_{ij} \ge 0 \quad \forall \ (i,j) \in A$$

$$(4.4.17)$$

$$\alpha_s = 0, \ \alpha_t = 1, \ \delta_{ts} = 0, \ \beta_{ts} = 0$$
 (4.4.18)

$$z \in (-\infty, \infty) \tag{4.4.19}$$

$$\alpha_i \in \{0,1\} \quad \forall \ i \in N \tag{4.4.20}$$

$$\beta_{ij}, \delta_{ij} \in \{0, 1\} \quad \forall \ (i, j) \in A'$$
(4.4.21)

where, O is the set of Benders' Decomposition optimality cuts, and γ_{ij}^{o} represents γ_{ij} values for the cut o. Equation 4.4.16 is the resource robustness constraint for the RNIP, written as a set of combinatorially many constraints instead of a single constraint with maximization, which was dualized to obtain the final RNIP model formulation (refer MODEL 2 in Section 3.4.1 or 3.4.2).

The motivation for the heuristic is to account for |O| constraints (in equation 4.4.15) using a single constraint. This would reduce the Master Problem to a Network Interdiction Problem with Resource Consumption Uncertainty only (NIPRCU). NIPRCU can be solved by first finding the min-cut of the network using representative capacities, and determining the arc set available for interdiction (A_I) . Then, determining δ (the interdiction decision) variables is a robust knapsack problem with arc capacities as item values, resource consumption as item weights, and Δ is the capacity of the knapsack (for solving the robust knapsack problem, refer Section 4.2). Once, δ variables are determined, the β variables can be easily determined, as they are forward arcs of the minimum cut which are not interdicted.

Calculating representative arc capacities of the network with respect to o^{th} optimality cut:

The method simultaneously penalizes the presence of optimality cuts in the set $O \setminus \{o\}$ with respect to the o^{th} optimality cut to calculate the representative arc capacities for the network. The penalty with respect to the o^{th} optimality cut

is calculated as:

$$cap_penalty^{o} = \begin{cases} \frac{\sum\limits_{(i,j)\in A'} \gamma_{ij}^{o} \hat{u}_{ij}}{\sum\limits_{(i,j)\in A'} \gamma_{ij}^{o}} & ; \text{ if } \sum\limits_{(i,j)\in A'} \gamma_{ij}^{o} > 0\\ 0 & ; \text{ otherwise} \end{cases}$$
(4.4.22)

The representative arc capacities of the network with respect to the o^{th} optimality cut (\tilde{u}^o) are then given as:

$$\tilde{u}_{ij}^{o} = \begin{cases} u_{ij} + \hat{u}_{ij} & ; \text{ if } \gamma_{ij}^{o} = 1 \\ u_{ij} & ; \text{ if } \gamma_{ij}^{o} = 0 \text{ and } \sum_{q \in O \setminus \{o\}} \gamma_{ij}^{q} = 0 \\ u_{ij} + \max\{0, \hat{u}_{ij} - cap_penalty^{o}\} & ; \text{ if } \gamma_{ij}^{o} = 0 \text{ and } \sum_{q \in O \setminus \{o\}} \gamma_{ij}^{q} > 0 \end{cases}$$
(4.4.23)

The above-mentioned penalty procedure guarantees the feasibility of the solution but not the optimality.

Steps for solving the Benders' Master Problem using the simultaneous penalty heuristic:

For each $o \in O$,

- 1. Calculate the representative arc capacities for the network (\tilde{u}^o) with respect to optimality cut o.
- 2. Determine the min-cut (N_s, N_t) of the network. The variable α^o and the

interdiction arc set (A_I) for the network are given as:

$$\alpha_i^o = \begin{cases} 1 & \text{; if } i \in N_t \\ 0 & \text{; if } i \in N_s \end{cases}$$
$$A_I = \left\{ (i,j) \in A' \mid \alpha_j^o - \alpha_i^o = 1 \text{ and } r_{ij} \le \Delta \text{ and } \gamma_{ij}^o = 0 \right\}$$

- 3. Initialize δ^o variables by solving the robust knapsack problem with parameters $c = \tilde{u}^o$, w = r, $\hat{w} = \hat{r}$, $b = \Delta$, $N = A_I$, and $\Gamma = \Pi$ (refer Section 4.2)
- 4. The value of ζ is the Π^{th} largest value of $\hat{r}_{ij}\delta^o_{ij}$.
- 5. The values of variables σ are given as: $\sigma_{ij} = \max\{0, \hat{r}_{ij} \zeta\}$
- 6. Determine the β^o variables as follows:

$$\beta_{ij}^{o} = \begin{cases} 1 & \text{; if } \alpha_{j}^{o} - \alpha_{i}^{o} = 1 \text{ and } \delta_{ij}^{o} = 1 \\ 0 & \text{; otherwise} \end{cases}$$
(4.4.24)

7. The value of z^o is $\sum_{(i,j)\in A'} \tilde{u}^o_{ij}\beta^o_{ij}$

The solution of the Master Problem is then given as:

 $z^* = \max(z^o)$ $\alpha^* = \{\alpha^o \mid z^o = z^*\}$ $\beta^* = \{\beta^o \mid z^o = z^*\}$ $\delta^* = \{\delta^o \mid z^o = z^*\}$ $\zeta^* = \{\zeta^o \mid z^o = z^*\}$ $\sigma^* = \{\sigma^o \mid z^o = z^*\}$

4.4.2 Solution procedure

The solution procedure followed here is the same as in Lasdon (2002), only that it has been modified to the context of RNIP.

- 1. Set tolerance value (default value, 10^{-4}), iteration limit value (default value, 20), maximum computational time (default value, 21600 sec), and $Z_{old}^{ub} = +\infty$.
- 2. Only for the first iteration, use the optimality cut with all γ_{ij} values as zero (this a trivial optimality cut, and it prevents unboundedness of Master Problem in the first iteration). The trivial optimality cut is removed for subsequent iterations as it would always be satisfied. Initialize $O = \{\}$.

- Solve the Master Problem using the heuristic described in Subsection 4.4.1 using the set O (except, in the first iteration), and determine the values: z*, α*, β*, δ*, ζ*, and σ*.
- 4. Solve Sub-Problem-Dual problem, using $\bar{\beta} = \beta^*$, and determine the value of γ^* (Optimal solution of dual of the Benders' sub-problem).
- 5. Calculate the bounds for the Benders' Decomposition as:

Lower Bound
$$(Z^{lb}) = z^*$$
 (4.4.25)

Upper Bound
$$(Z^{ub}) = \sum_{(i,j)\in A'} (u_{ij} + \gamma^*_{ij}\hat{u}_{ij})\beta^*_{ij}$$
 (4.4.26)

6. Check if the current upper bound is better than the previous best upper bound, and update the current best solution accordingly. That is:

(a) if
$$Z^{ub} < Z^{ub}_{old}$$
: do
$$Z^{ub}_{old} = Z^{ub}.$$

Update the current best solution with $\alpha^*, \, \beta^*, \, \delta^*, \, \gamma^*, \, \zeta^*$, and σ^*

(b) else: do

$$Z^{ub} = Z^{ub}_{old}$$

7. Calculate the MIPGap as:

$$MIPGap = \frac{Z^{ub} - Z^{lb}}{Z^{ub}} \tag{4.4.27}$$

- 8. If *MIPGap* is less than the tolerance, then the algorithm has converged.
- 9. If *MIPGap* is greater than the tolerance then,
 - (a) Check if the γ* value found in the current iteration is different from the previously calculated values of γ* (i.e., the γ* values stored in the set O).
 - (b) If the current iteration's γ^* value is different from the previously calculated ones, then, add the following cut to the Master Problem:

$$\left(\sum_{(i,j)\in A'} (u_{ij} + \hat{u}_{ij}\gamma^*_{ij})\beta_{ij}\right) - z \le 0$$

- (c) If the value of γ^* calculated in the current iteration was equal to one of the previous optimality cuts, then, terminate the algorithm with the current best solution.
- 10. If the iteration limit is reached or the computational time is exceeded, terminate the algorithm.
- 11. Continue Steps 3-10 until one of the termination criteria is satisfied.

The Benders' Decomposition framework used here does not guarantee solution optimality as the Master Problem is not solved exactly but using a heuristic. As the solution of the Master Problem affects the lower bound in the context of RNIP, it should be noted that the lower bound reported by Benders' Decomposition, here, is an upper bound solution of the real lower bound. Therefore, the MIPGap values reported are an underestimate of the actual MIPGap values.

4.5 Enhanced Benders' Decomposition-based Heuristic

The idea of the current heuristic is to merge the best qualities of Lagrangian Relaxation and Benders' Decomposition into one unified heuristic. For the heuristic, Benders' Decomposition is preceded by Lagrangian Relaxation. The Lagrangian Relaxation provides with the first optimality cut, and better upper and lower bounds, after which Benders' Decomposition is initiated. The procedure for the heuristic is as follows:

- 1. Solve the RNIP problem using Lagrangian Relaxation with an iteration limit (default value, 10), tolerance (default value, 10⁻⁴), and maximum computational time (default value, 21600 sec). The procedure for Lagrangian Relaxation is detailed in Section 4.3. If the Lagrangian Relaxation terminates with the tolerance criteria, the optimal solution is found, and the procedure is terminated.
- 2. Using the feasible β variable values (i.e. β^{LR} , the upper bound solution of Lagrangian Relaxation), determine the first Benders' Decomposition

optimality cut, by solving the following to obtain γ^* :

$$\max_{\gamma} \sum_{(i,j)\in A'} \hat{u}_{ij} \beta_{ij}^{LR} \gamma_{ij}$$
$$\sum_{(i,j)\in A'} \gamma_{ij} \leq \Gamma$$
$$\gamma_{ij} \leq 1 \quad \forall \ (i,j) \in A'$$
$$\gamma_{ij} \geq 0 \quad \forall \ (i,j) \in A'$$

The above problem is a 0-1 nominal knapsack problem with all weights of all γ variables as 1, for non-negative integer values of Γ . The solution to the above problem is selecting the top Γ arcs with respect to values $\hat{u}_{ij}\beta_{ij}^{LR}$.

3. Initialize the optimality cut set (O) with the following constraint:

$$\left(\sum_{(i,j)\in A'} (u_{ij} + \hat{u}_{ij}\gamma_{ij}^*)\beta_{ij}\right) - z \le 0$$

4. Initialize the Z_{old}^{ub} for Benders' Decomposition with the Lagrangian Relaxation Upper Bound and the current best solution with the Lagrangian Relaxation upper bound solution. Set Z_{old}^{lb} for Benders' Decomposition with the Lagrangian Relaxation lower bound value. Set tolerance value (default value, 10^{-4}) and iteration limit (default value, 20) for Benders' Decomposition.

- Solve Benders' Decomposition by following steps 3-10 detailed in Section
 4.4.2. Add an additional step to retain the best lower bound value in each
 Benders' Decomposition iteration.
- 6. Once Benders' Decomposition has terminated, report the best upper bound, best solution, and best lower bound values.

Like Benders' Decomposition, the enhanced Benders' Decomposition also provides an upper bound solution of Master Problem. This causes the estimated lower bound to be higher than the actual lower bound, resulting in the reported MIP gaps which are smaller than the actual gap.

4.6 Summary

This chapter presented three solution heuristics for the RNIP in the context of illicit drugs. The first heuristic was developed based on Lagrangian Relaxation. After relaxing three constraints in RNIP (MODEL 2 formulation), the problem decomposed into a maximum flow problem and four unconstrained optimization problems. Valid upper bounds were derived for the variables in the unconstrained optimization sub-problems to prevent unboundedness. The feasible solution for the problem was developed by exploiting the fact that robustness and interdiction decisions only depend on the minimum cut of the network, and robust knapsacks were used to determine the arcs for interdiction.

The second heuristic was developed based on Benders' Decomposition. Only the capacity robustness variables were partitioned out of the RNIP so that the Benders' sub-problem would only result in optimality cuts. This is desired as it allows for the determination of the MIP gap after every iteration. It was shown that the computational complexity of the Benders' Master Problem was the same as that of RNIP, and therefore a heuristic was designed to obtain a strong upper bound on the Master Problem by using a simultaneous penalty approach. The simultaneous penalty method aims to condense all the capacity-related optimality cuts into a single optimality cut so that the RNIP consists only of the resource consumption robustness, which can be solved exactly. Finally, the third heuristic, enhanced Benders' Decomposition, aimed at unifying the Lagrangian Relaxation framework and the Benders' Decomposition framework, with the hopes of achieving better solution confidence. For realizing this, the Lagrangian Relaxation was first performed. The Lagrangian Relaxation best solution was used to determine the initial optimality cut for Benders' Decomposition, and the Lagrangian Relaxation bounds were used to initialize the bounds for Benders' Decomposition.

The next chapter discusses the computational efficiency of the above algorithms with respect to solving RNIP using a MIP solver. Sensitivity analyses are performed to evaluate the effect of changes in the robustness control parameters, and changes in magnitude of uncertainty considered. An analysis is performed to evaluate the value of considering uncertainty in the decision-making process.

5 Analysis

Robust Network Interdiction is carried out on artificial test networks that represent drug transportation through roads or rivers. The network design is similar to the one presented in Bingol (2001). The network is considered as a $n_1 \times n_2$ grid, where n_1 is the number of nodes on the vertical axis and the n_2 is the number of nodes horizontal axis. The source is connected to all the 'westernmost' nodes by non-interdictory arcs of infinite capacity. Similarly, all the 'easternmost' nodes are connected to the sink by non-interdictory arcs of infinite capacity. An artificial infinite-capacity non-interdictory arc from the sink node to the source node is constructed to make modeling simpler. Within the $n_1 \times n_2$ grid, all nodes on the horizontal axes are connected by 'west-to-east' arcs. All the nodes on the vertical axes are connected by either 'north-to-south' or 'south-to-north' arcs which keep alternating in both directions, with the 'north-western' most vertical arc in 'north-to-south' direction. The whole $n_1 \times n_2$ grid has diagonal arcs flowing in 'north-east' and 'south-east' directions. For all interdictory arcs, the nominal capacities are integers varying between 10 and 100 units. The capacity variance of each arc is an integer between 10% and 30% of the arc's capacity. The parameters Γ , Π and Δ are set at 20 units, 2 units and 2000 units, respectively. For all interdictory arcs, r_{ij} is assumed as 100 units, and for non-interdictory arcs, it is

set at 10^6 units. The \hat{r}_{ij} values are randomly chosen to be an integer between 10 and 30 units. For all infinite capacity non-interdictory arcs, both the arc capacity and resource consumption variances are set to zero. The general network is represented as $n_1 \times n_2$ further on. Table 5.1 gives a summary of the number of nodes and directed arcs (including the artificial return arc) in the network, and Figure 5.1 visually represents 4×4 . All the computational analyses are carried out on 10 test networks for the sizes of 50×50 , 100×100 , and 200×200 generated as per the aforementioned procedure. Only a single test network of size 500×500 is tested for all the following analyses.

Table 5.1: Test Networks

Network	Number of nodes	Number of directed arcs
50×50	2502	9803
100×100	10002	39603
200×200	40002	159203
500×500	250002	998003

All the computational experiments are carried out on one of the two Windows desktops with Intel core i7 processors, 4 cores, 8 logical processors, and 32 GB of RAM. The MIP solver used in the current study is Gurobi Optimizer 8.1 [Gurobi Optimization (2019)] (referred to as 'Gurobi' for the rest of the document).



Figure 5.1: Visual representation of 4×4

5.1 Computational Efficiency

The computational efficiency of the RNIP can affect the decisions in real-life, as most of the times the tactical intelligence received is time-sensitive [Steinrauf (1991), Kenney (2003), Meyer and Anderson (2008)]. In this section, the three heuristics developed in the study are compared to the solutions obtained using a state-of-the-art MIP solver. Later sections cover sensitivity analysis for changes in the robustness control parameters (i.e., the budgets of uncertainty) and changes in the amount of uncertainty in the data. The test networks are solved using:

Gurobi optimizer in Python interface. The model is run for a maximum of 21600 sec for 50×50, 100×100, and 200×200, and 43200 sec for 500×500. The default tolerance of 10⁻⁴ is used. All other parameters are also used at their default values.

- Lagrangian Relaxation (LR) implemented in Python. The computational time limit is set to 21600 sec for all the 31 test networks, and a tolerance value of 10⁻⁴ is used. An iteration limit of 20 is used for all test networks.
- Benders' Decomposition (BD) implemented in Python. The computational time limit is set to 21600 sec, the tolerance value is set to 10⁻⁴, and the iteration limit is set to 20 for all the 31 test networks.
- The Enhanced Benders' Decomposition (EBD) implemented in Python.
 The computational time limit is set to 21600 sec, and the tolerance value is set to 10⁻⁴. The LR is run for a maximum of 10 iterations, followed by a maximum of 20 iterations for BD.

All the instances of solving a maximum flow problem (or minimum cut problem) in LR or BD are solved using the push-relabel maximum flow algorithm available in the *networkx* library in Python [Hagberg et al. (2008)]. The push-relabel maximum flow algorithm was developed by Goldberg and Tarjan (1988), and is one of the most efficient algorithms for solving maximum flow problems with a strongly polynomial time complexity of $O(|N|^2 \cdot |A|)$. The nominal 0-1 maximum profit knapsack problems while solving the Robust Knapsack Problem (refer Section 4.2) are solved by modeling them as an optimization problem in Gurobi as it was found to be computationally efficient than dynamic programming. Default values of 20, 2, and 2000 are used for Γ , Π , and Δ , respectively. Table 5.2 summarizes the computational performance of Gurobi and the three developed heuristics for solving RNIP. It can be noted that Gurobi is unable to find an initial solution for the test network 500×500 even after 12 hours of computation. For all 50×50 test networks, Gurobi found an optimal solution. Gurobi found an optimal solution for only one of the 10 test networks of sizes 100×100 and 200×200 . For the rest of the cases, Gurobi terminated using the maximum computational time criteria. All cases of LR terminated using the maximum iterations criteria, while all cases of BD and EBD terminated because no new optimality cuts could be found.



■ Lagrangian Relaxation ■ Benders' Decomposition ■ Enhanced Benders' Decomposition



Figure 5.2 shows the computational time savings achieved by using LR, BD, and EBD for solving RNIP with respect to the time required by Gurobi. The overall time savings is defined as the ratio of total time saved while solving all the 10 test networks of the same size by a heuristic in comparison to Gurobi to the total time required to solve the same using Gurobi. The overall time savings while using LR are 86.75% (minimum: -204.75%, maximum: 95.74%) for 50×50 networks, 98.63% (minimum: -10.28%, maximum: 99.09%) for 100×100 networks, and 93.3% (minimum: 86.8%, maximum: 95.63%) for 200×200 networks when compared to Gurobi. The overall time savings in the case of using BD to solve RNIP compared to Gurobi are: 92.66% (minimum: -48.15%, maximum: 97.82%), 99.33% (minimum: 71.94%, maximum: 99.66%), and 97.23% (minimum: 96.32%, maximum: 98.21%) for test networks of sizes 50×50 , 100×100 , and 200×200 , respectively. For EBD, the overall time savings are: 90.52% (minimum: -195.24%, maximum: 97.26%), 99.33% (minimum: 17.79%, maximum: 99.21%), and 94.78% (minimum: 91.19%, maximum: 96.34%) for test networks of sizes 50×50 , 100×100 , and 200×200 , respectively. It is interesting to note that the time taken by Gurobi to compute the first solution for any 200×200 network is more than twice the total time required by any of the three heuristics developed in this study.



Figure 5.3: Maximum final gap values while solving RNIP

Figure 5.3 shows the maximum final gap values achieved by Gurobi, LR, BD, and EBD, and Figure 5.4 shows the maximum deviation of upper bound achieved by LR, BD, and EBD from the Gurobi upper bound. The final gaps while solving RNIP using Gurobi were a maximum of 0.01%, 0.32%, and 0.4% for the 50×50, 100×100, and 200×200 networks, respectively. The similar values of LR were 13.17%, 5.29%, and 2.32% respectively. In spite of large final gap values for LR, their final upper bounds found were no further than 1.44%, 0.32%, and 0.12% from the maximum flow found using Gurobi for the 50×50, 100×100, and 200×200 networks, respectively. So, even though the confidence in the solution is lower, the quality of the solution is very strong. The maximum final gap values for BD are 5.13%, 3.23%, and 1.58% for network sizes 50×50 , 100×100 , and 200×200 , respectively. The maximum deviation in upper bounds found using BD from the Gurobi upper bounds are 2.04%, 0.36%, and 0.12% for the test networks 50×50 , 100×100 , and 200×200 , respectively. On average, the solution quality found using the BD is worse than LR, however, the confidence on the solution found is much better. EBD results in the maximum final gap values of 4.58%, 3.01%, and 1.48% respectively for the network sizes 50×50 , 100×100 , and 200×200 , making them better than the BD counterpart values. The maximum deviation in the upper bounds found using EBD from the Gurobi upper bounds are the same as reported for LR. EBD is a true amalgam of LR and BD, giving the strong solution quality of LR while improving on the already superior confidence on solution obtained using BD, imbibing the best qualities from both the previous approaches.



■ Lagrangian Relaxation ■ Benders' Decomposition ■ Enhanced Benders' Decomposition

Figure 5.4: Maximum deviation from Gurobi upper bound solution
		50×50				100×100			200×200		500×500
		Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	500×500
Final UB w.r.t.	LR	100	100.57	101.44	100.07	100.19	100.32	100.02	100.06	100.12	NA
Final Gurobi	BD	100	100.96	102.04	100.07	100.23	100.36	100.03	100.08	100.12	NA
UB (%)	EBD	100	100.57	101.44	100.07	100.18	100.3	100.02	100.06	100.12	NA
	Gurobi	0	0	0.01	0	0.2	0.32	0	0.26	0.4	NA
Final Gap (%)	LR	5.9	10.05	13.17	2.48	3.91	5.29	1.48	1.85	2.32	0.93
	BD	0.49	2.55	5.13	0.94	2.28	3.23	0.56	1.11	1.58	0.51
	EBD	-0.16	1.96	4.58	0.61	2.15	3.01	0.46	1.06	1.48	0.51
	Gurobi	21	476	1454	253	19465	21600	7886	20229	21600	NA
Final Run Time	LR	61	63	65	196	267	305	945	1356	1552	6418
(sec)	BD	10	35	118	71	130	322	194	560	795	2898
	EBD	34	45	64	170	234	428	656	1055	1904	5683
	Gurobi	8	9	10	159	181	196	4035	4245	4473	NA
Time to 1st	LR	3	3	3	14	14	15	64	70	79	663
solution (sec)	BD	2	3	3	10	11	12	46	49	55	416
	EBD	3	3	3	14	14	15	64	70	79	663

Table 5.2: Performance of Gurobi and heuristics for RNIP

(a) LR: Lagrangian Relaxation; BD: Benders' Decomposition; EBD: Enhanced BD, UB: Upper Bound.

(b) NA: Not Applicable. Gurobi could not provide any solution for the test network 500×500 .

(c) Percentage and time values have been rounded to nearest hundredths and ones, respectively.

The quality of the first upper bound found by the heuristics is examined to check the solution stability, as well as their applicability in very time-sensitive scenarios. It was found the first upper bound reported by all the three heuristics is the same. As the procedure for EBD starts with LR, their first upper bounds should be the same. BD starts by solving a network interdiction model with resource consumption uncertainty only, and then adds capacity uncertainty to determine the first solution. LR determines the minimum cut using nominal capacity values (because of the chosen initial values of the Lagrangian parameters) and solves for the interdiction decision considering resource consumption uncertainty, following which it incorporates capacity uncertainty decision to determine the first solution, essentially making it the same as the first problem encountered in BD. Therefore, all three heuristics have the same first upper bound. Figure 5.5 depicts the variation in the quality of the first upper bound with network size. The maximum deviation of the heuristics' first upper bound from the Gurobi final upper bound found is 2.06%, 0.56%, and 0.12% for test networks of sizes 50×50 , 100×100 , and 200×200 , respectively. Also, all heuristics require less than 100 sec to arrive at the first solution with network size as large as 200×200 .

Though time savings using EBD are less than BD, it does provide better solution quality as well as solution confidence than BD. Smith and Song (2019) cite the computational inefficiency of current algorithms as one of the major bottlenecks faced to determine strategies in network interdiction models. All the

96



Figure 5.5: Quality of the first upper bound determined by heuristics

developed heuristics are time-efficient and result in improved solution quality and confidence as the network size increases. Among the three heuristics, EBD provides a near-perfect compromise between solution quality, solution confidence, and computational time, and therefore, EBD is chosen to conduct further analyses on the sensitivity of the model to robustness parameters, and amount of uncertainty in parameter estimation. Finally, an analysis for determining the value of considering uncertainty in decision-making is conducted.

5.2 Sensitivity to Changes in Amount of Uncertainty in Parameters

When the illicit drug transportation network is modeled, the decision-maker (local enforcement agency) does not have complete knowledge of the transportation network. Using the current intelligence, local surveillance patterns, and drug avail-

ability at the destination market, the local agency tries to determine the nominal capacity of the link in the network, the nominal resource consumption required for interdiction, and their variances from the nominal values. The confidence in the estimated variances for the capacity of the arc and the resource consumption depends on the quality of information available at the time of decision-making, motivating the sensitivity analysis for the amount of uncertainty in the network parameters. A Monte Carlo scheme is adopted to simulate actual realizations of the illicit drug transportation network, and the performance of the interdiction decisions are studied. The procedure adopted to set up the Monte Carlo simulations is given in algorithm 1. The network is modeled at three levels of uncertainty: Low, where \hat{u} and \hat{r} vary between 10% and 30% of u and r, respectively (the base case); Moderate, where the variances are between the 20% and 60% of the nominal values; and, High, where the variances are between the 30%and 90% of the nominal values. The sensitivity analysis is conducted by solving RNIP using EBD (as it provided the best compromise, see section 5.1), using the base case values of Γ , Π , and Δ as 20, 2, and 2000, respectively. A summary of the sensitivity analysis is presented in Table 5.3. The calculated maximum flow refers to the robust maximum flow found using EBD. Interdiction success is the ratio of total_success and total_attempts calculated after the termination of Monte Carlo scheme (Algorithm 1). The minimum increase, average increase, and maximum increase in the actual maximum flow refer to the variation across the

different Monte Carlo simulations for a single network. The increases are calculated with respect to the calculated maximum flow. The 'Min', 'Ave' and 'Max' associated with a network size refer to the variation across the 10 different test networks of the same size. The step random.seed(0) ensures that a test network is tested using the same actual realizations in the Monte Carlo simulation independent of the algorithm used to determine the interdiction decision. The command $networkx.maximum_flow_value(G, s, t)$ computes the maximum flow in a directed graph G from source node s to the sink node t.

From table 5.3, it can be noticed that the calculated maximum flow increases with an increase in the amount of uncertainty in the estimation of arc capacity and resource consumption, showing that robust model is sensitive to the amount of variation. An interdiction attempt is successful only if the allocated resources are greater or equal to the resource requirement. Because of the way the interdiction success criterion is designed, and the way the actual values are calculated, the interdiction success rate is independent of the amount of uncertainty. The negative values of the increases in the actual maximum flows with respect to the calculated robust maximum flows represent that the calculated maximum flow overestimates the actual value and vice versa. The average increase in the actual maximum flow values indicates that the robust model always underestimates the actual maximum flow when the uncertainty is low, always overestimates the actual maximum flow when uncertainty is high and may underestimate or overestimate the actual maximum flow by a small amount when the uncertainty is moderate. The overestimation is preferred to underestimation as it is better if the actual illicit drug flow to the destination market is lower than what is expected. The robust model performs better with the same budgets of robustness when the amount of uncertainty is high rather than low, which seems counter-intuitive. This is because the budget of capacity uncertainty (Γ) can provide more protection at a higher amount of certainty than at a lower amount of uncertainty. As a low amount of uncertainty presents a greater challenge, further analyses are performed using low uncertainty in network parameters. The next section discusses the effect of different budgets of uncertainty on actual maximum flows.

Algorithm 1 Monte Carlo simulation scheme

Solve the network interdiction model with parameters u, \hat{u}, r and \hat{r} and determine the optimum values of decision variables: δ^* and π^* The interdiction decision set, $A_{IA} = \{(i, j) \mid \delta_{ij}^* = 1 \ \forall (i, j) \in A'\}$ Resource allocation, $\breve{r} = \{(r_{ij} + \pi_{ij}^* \hat{r}_{ij} \forall (i,j) \in A_{IA}; 0 \forall (i,j) \in A' \setminus A_{IA}\}$ $total_attempts = 0; total_success = 0$ g = 1000 for 50×50 , 100×100 ; g = 200 for 200×200 ; g = 100 for 500×500 $current_iter = 0; MCSim_iter = q$ $actual_max_flows = zeros(MCSim_iter)$ random.seed(0)while *current_iter* < *MCSim_iter* do Generate actual arc capacity, $\tilde{u}_{ij} = Uniform(u_{ij} - \hat{u}_{ij}, u_{ij} + \hat{u}_{ij})$ Generate actual resource requirement, $\tilde{r}_{ij} = Uniform(r_{ij} - \hat{r}_{ij}, r_{ij} + \hat{r}_{ij})$ for $(i, j) \in A_{IA}$ do $total_attempts + = 1$ if $\check{r}_{ij} \geq \tilde{r}_{ij}$ then $total_success + = 1$ $\tilde{u}_{ij} = 0$ end if end for Let G be the directed graph with capacities \tilde{u} Let s be the source node, and t be the sink node $actual_max_flows[current_iter] = networkx.maximum_flow_value(G, s, t)$ $current_iter + = 1$ end while

	Amount of		50×50			100×100)		F00×F00		
	uncertainty	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	900 × 900
Calculated	Low	4096	4378	4566	9758	10157	10404	21253	21802	22019	57231
maximum flow	Moderate	4483	4781	4927	10449	10630	10769	21915	22211	22536	57693
(units)	High	4870	4995	5143	10723	10990	11237	22041	22695	22996	58243
Intendiction	Low	55.69	56.59	57.09	55.76	56.56	57.23	54.71	56.59	58.53	56.89
Interdiction success rate $(\%)$	Moderate	55.56	56.19	57.22	55.86	56.23	56.64	54.82	55.62	56.24	53.95
	High	55.31	55.88	56.47	55.4	55.87	56.54	54.22	55.78	58.06	56.22
Minimum	Low	-3.98	-2.53	-1.46	-3.58	-2.51	-1.62	-2.96	-1.78	-0.91	-1.32
increase in	Moderate	-14.07	-11.27	-9.46	-9.95	-8.47	-7.22	-7.64	-6.14	-5.17	-4.72
AMF $(\%)$	High	-21.14	-18.86	-14.62	-18.33	-15.54	-13.72	-13.64	-12.85	-11.67	-11.45
Average	Low	8.16	9.32	10.73	3.15	3.46	3.91	0.65	1.02	1.47	-0.32
increase in	Moderate	-0.05	1.12	2.44	-2.85	-1.85	-1.03	-3.46	-3.06	-2.58	-3.59
AMF $(\%)$	High	-7.77	-6.7	-5.52	-9.86	-8.96	-8.16	-10.25	-9.41	-8.9	-9.98
Maximum	Low	19.63	21.76	24.5	8.4	9.33	10.29	2.95	3.61	4.39	0.58
increase in	Moderate	10.42	13.75	18.74	2.92	3.88	4.78	-0.68	-0.08	0.69	-2.79
AMF $(\%)$	High	2.92	4.5	5.74	-5.25	-3.92	-2.33	-7.72	-6.87	-6.03	-9.03

Table 5.3: Sensitivity to changes in the amount of uncertainty of parameters

(a) AMF: Maximum flow found in a Monte Carlo Simulation.

(b) Increases are with respect to the calculated maximum flow for the test network.

(c) All percentage values are rounded to the nearest hundredths

5.3 Sensitivity to Changes in Budgets of Uncertainty

The budgets of uncertainty help control the amount of uncertainty incorporated into modeling and therefore, the conservativeness of the solution obtained. Therefore, a sensitivity analysis on the budgets of uncertainty is conducted to find robustness parameters which provide explain the actual results the best. For the same, the RNIP is solved using EBD (as it provided the best compromise, refer section 5.1) for the case of 'Low' amount of uncertainty in the estimation of arc capacity and resource consumption for interdiction (as it proved to be the most challenging, refer section 5.2). The case of 'Low' uncertainty provided an underestimate of the actual maximum flows on average, indicating that budgets of uncertainty should be increased to provide a better estimate of the actual maximum flows in the network. Firstly, the sensitivity analysis on the budget of capacity uncertainty (Γ) is conducted by varying it for the values of 20 (the base case), 30, and 40, while maintaining the values of Π and Δ at their base case values of 2 and 2000. Secondly, the sensitivity analysis is performed on the budget of resource consumption uncertainty (Π) by varying it to the values of 2 (the base case), 6, and 10, while adopting the base case values of 20 and 2000 for the parameters Γ and Δ . The Monte Carlo scheme adopted in section 5.2 is used to measure the changes in actual maximum flow while changing the robustness control parameters Γ and Π .

			50×50			100×100)		200×200)	500, 500
		Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	200×200
Calculated	20	4096	4378	4566	9758	10157	10404	21253	21802	22019	57231
maximum flow	30	4153	4466	4659	9887	10284	10548	21417	21974	22189	57458
(units)	40	4230	4542	4740	10010	10402	10629	21577	22135	22353	57673
Interdiction	20	55.69	56.59	57.09	55.76	56.56	57.23	54.71	56.59	58.53	56.89
success rate	30	56.01	56.52	57.16	55.75	56.49	57.07	54.95	56.63	58.53	56.89
(%)	40	56.01	56.64	57.16	55.52	56.49	57.07	54.82	56.48	58.53	56.89
М	20	3984	4268	4426	9424	9902	10152	20824	21415	21708	56476
Minimum	30	3964	4273	4426	9551	9907	10152	20832	21406	21744	56476
AMF (units)	40	3980	4275	4426	9538	9908	10197	20792	21417	21744	56476
A	20	4504	4786	4938	10140	10508	10731	21565	22025	22265	57048
Average	30	4507	4785	4938	10141	10510	10737	21554	22031	22268	57048
AMF (units)	40	4508	4787	4938	10139	10510	10741	21581	22028	22268	57048
M	20	4993	5330	5511	10762	11104	11345	22186	22590	22896	57562
ME(unita)	30	5162	5364	5511	10762	11120	11395	22087	22565	22878	57562
$\mathrm{AMF}(\mathrm{units})$	40	5034	5340	5511	10807	11109	11340	22127	22571	22878	57562

Table 5.4: Sensitivity to changes in the budget of capacity uncertainty

(a) AMF: Actual Maximum flow found in Monte Carlo Simulation.

(b) Interdiction success rates and flow values are rounded to the nearest hundredths and ones, respectively.

Table 5.4 summarizes the effect of changing Γ on the actual maximum flows. The calculated maximum flow refers to the robust maximum flow found using EBD. Interdiction success rate is the ratio of *total_success* and *total_attempts* calculated after the termination of Monte Carlo scheme (Algorithm 1). The minimum, average, and maximum actual maximum flows refer to the variation across the different Monte Carlo simulations for a single network. The 'Min', 'Ave' and 'Max' associated with a network size refer to the variation across the 10 different test networks of the same size. The effect of an increase in Γ increases the calculated maximum flow by about 2% for 50×50 test networks per 10 unit increase in Γ . The increase in calculated maximum flow for 100×100 , 200×200 , and 500×500 per 10 unit increase in Γ is about 1.3%, 0.8%, and 0.4%, respectively. This is expected as the conservativeness of the solution increases with an increase in the budget of uncertainty. However, changes in Γ do not affect interdiction success, and the actual flow at all. This occurs because the change in Γ affects the calculated maximum flow after the interdiction decisions are made, and helps decide how much of the variation in arc capacity is accepted. The best outcomes were achieved using the Γ value of 40 which caps the deviation of the average actual maximum flow from the calculated maximum flow at 6.83% underestimation of actual flow.

Table 5.5 summarizes the effect of changing Π on the actual maximum flows. The calculated maximum flow increases with an increase in Π values on average for all network sizes. This is expected because, with an increase in the budget of uncertainty, the conservativeness of the estimate should increase. As Π accounts for uncertainty in resource consumption, it also affects the interdiction decisions. Therefore, with an increased accounting of variation in resource consumption, the chances of a successful interdiction should increase, as can be seen in the table 5.5. For the actual maximum flow calculations, there are two opposing effects with increasing values of Π : the actual maximum flow should increase as the number of interdiction attempts decrease as we account for more uncertainty in resource consumption, and the actual maximum flow should decrease as the chances of a successful interdiction increases with increasing values of Π . It can be noticed that the effect of the second factor (increased interdiction success rate) dominates the effect of the first factor (reduced interdiction attempts), resulting in a strong decline in actual maximum flows with an increase in the values of Π . The Π value of 10 achieves the best outcomes, with the average and the absolute deviation of the average actual maximum flow from the calculated maximum flow being less than 1% and 2%, respectively, overall the test networks.

The comparison of the best performing Γ , Π combinations of 40,2 and 20,10 show the latter combination provided better results as the deviation of the actual maximum flow from the calculated maximum flow is lesser and the interdiction success rates are higher than the prior Γ , Π combination.

	<u>п</u>		50×50			100×100)		200×200)	500, 500
		Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	000×000
Calculated	2	4096	4378	4566	9758	10157	10404	21253	21802	22019	57231
maximum flow	6	4114	4441	4636	9841	10233	10471	21348	21891	22111	57265
(units)	10	4152	4480	4678	9914	10289	10489	21380	21929	22151	57329
Interdiction success rate (%)	2	55.69	56.59	57.09	55.76	56.56	57.23	54.71	56.59	58.53	56.89
	6	67.31	67.73	68.21	66.92	67.65	68.01	65.97	67.61	69.44	66.63
	10	78.53	78.87	79.98	78.04	78.7	80.46	77.67	78.6	79.47	78.22
	2	3984	4268	4426	9424	9902	10152	20824	21415	21708	56476
Minimum AME (upita)	6	3912	4208	4346	9509	9851	10089	20676	21361	21710	56389
AMF (units)	10	3872	4151	4320	9410	9805	9986	20698	21262	21526	56310
A	2	4504	4786	4938	10140	10508	10731	21565	22025	22265	57048
Average	6	4356	4659	4817	10003	10377	10607	21439	21895	22126	56924
AMF (units)	10	4198	4508	4692	9852	10222	10442	21253	21738	21967	56768
N/	2	4993	5330	5511	10762	11104	11345	22186	22590	22896	57562
Maximum AMF(units)	6	4841	5122	5266	10601	10911	11115	22061	22421	22766	57533
	10	4553	4871	5034	10253	10655	10886	21695	22186	22480	57333

Table 5.5: Sensitivity to changes in the budget of resource consumption uncertainty

(a) AMF: Actual Maximum flow found in Monte Carlo Simulation.

(b) Interdiction success rates and flow values are rounded to the nearest hundredths and ones, respectively.

5.4 Value of Adding Robustness

This section determines the value of considering uncertainty in the decisionmaking process of network interdiction, by comparing it with scenarios not considering the uncertainty (i.e. the deterministic network interdiction problem, or DNIP). The differences in the interdiction decisions when uncertainty is considered versus when uncertainty is not considered are apparent even for trivial small networks (refer example presented in section 1.1). For an additional example, consider a 4×4 test network generated as per the procedure mentioned at the very beginning of this chapter and adopting the values of Γ , Π , Δ as 2, 1, and 220. The interdiction decisions when the network is solved using RNIP versus DNIP are shown in figure 5.6. It can be noticed that incorporating capacity robustness can change the minimum cut of the network, and along with resource consumption robustness it can affect the interdiction decisions.

For detailed analysis, the test networks for the case of 'Low' uncertainty (variances are 10%-30% of the nominal values of arc capacity, and resource consumption) as it proved to be the most challenging (refer to section 5.2). Firstly, the test networks are solved as a DNIP using Gurobi. The Gurobi termination criteria are tolerance of 10^{-4} or maximum computational time of 21600 sec (the only exception is the 500×500 test network, for which the time limit is 43200 sec). The value of parameter Δ is set to 2000. The solutions of DNIP are compared to



(b) DNIP

Figure 5.6: Minimum cut and the interdiction decisions of a 2×2 test network



Difference between AMF and CMF — Probability of interdiction success

Figure 5.7: Sensitivity to changes in Γ ($\Pi = 2, \Delta = 2000$)

those obtained by solving the network as an RNIP using EBD.

Figure 5.7 shows the performance of RNIP with varying Γ values (with $\Pi = 2$) and DNIP on 100×100 test networks. It can be noticed that the robust models provide a better estimate of the actual flows and the probability of interdiction success improves by about 5%. The best performing Γ , Π combination here is 40, 2. Figure 5.8 shows the performance of RNIP with varying Π values (with $\Gamma = 20$) and DNIP on 100×100 test networks. The robust model provides a better estimate of the actual maximum flow than the deterministic model here as well. Also, the probability of interdiction success increases with an increasing value of Π , reaching almost 80% interdiction success at $\Pi = 10$. The Γ , Π combination of 20,10 outperforms the previous best combination of 40,2.

RNIP is solved using EBD using two Γ , Π combinations are used: first, the base case combination of 20,2; second, the best performing combination of 20,10.





Figure 5.8: Sensitivity to changes in Π ($\Gamma = 20, \Delta = 2000$)

When RNIP is solved using the budgets of uncertainty as 0, it reduces to a DNIP (for a detailed explanation, refer section 3.5). Therefore, DNIP can be represented as an RNIP with a Γ , Π combination of 0,0. However, the formulation proposed by Wood (1993) is used to solve DNIP, as it does not have any redundant constraints which occur in RNIP when the Γ , Π combination of 0,0 is used simply. The value of parameter Δ is set as 2000 for both RNIP and DNIP. After solving the test networks as a DNIP or RNIP, a Monte Carlo scheme is adopted to evaluate actual maximum flows and the interdiction success rates. The procedure used for Monte Carlo simulation is the same as one adopted in Section 5.2.

Table 5.6 presents a summary of the comparison between outcomes obtained by solving the network interdiction problem in a deterministic manner versus a robust manner. It can be observed the conservativeness of the calculated solution increases sharply when uncertainty in arc capacity and resource consumption is

accounted for. The interdiction success rate increases by about 5% on average for the base case Γ , Π combination of 20,2, and about 27% on average for the best performing Γ , Π combination of 20,10, when compared to the deterministic counterpart. The average actual flows for the robust solution with the Γ value of 20 and the Π value of 2 are only just slightly better than the deterministic solution. Even though the actual maximum flows are the same, the deviation from the calculated maximum flows is lower for the Γ , Π combination of 20,2 than the combination 0.0. However, with the Γ value of 20 and the Π value of 10, the actual maximum flows reduce significantly, as much as 6.8% on average for test networks of size 50×50 compared to the case of deterministic solution. Also, the actual maximum flows deviate by at most 1% only from the calculated maximum flow on average for the Γ , Π combination of 20,10 making the estimates very reliable. The experiment shows that incorporating robustness in decision-making can improve the final outcomes even though it provides more conservative solutions. However, these improvements may seem practically negligible considering the added efforts required to measure and incorporate uncertainty into modeling. One of the reasons why improvements are meager is that the interdiction attempt made on the network is relatively smaller. The value of parameter Δ at 2000, represents interdiction attempt on only about 14%, 7%, and 3.5% of the forward flowing arcs in the minimum cut of 50×50 , 100×100 , and 200×200 test networks, respectively. For the next experiment, larger interdiction attempts are

made with greater budgets of uncertainty with a hypothesis that the robust model would perform significantly better than than the deterministic model and lead to pragmatically considerable reductions in maximum flow.

For the evaluation of the value of adding robustness while undertaking a larger interdiction attempt, two values of parameter Δ are considered: 10000 and 15000. The robust model is solved using EBD with the Γ, Π combinations of 60,60 and 100,100 when Δ values are 10000 and 15000, respectively. Table 5.7 summarizes the results obtained using 10000 as the value of parameter Δ . The Δ value of 10000 represents an interdiction attempt on about 68%, 34%, and 17%of the forward arcs of the minimum cut of 50×50 , 100×100 , and 200×200 test networks in the deterministic scenario, respectively. It can be noticed that the difference in the probability of interdiction success is about 34% across all network sizes. The robust model provides highly conservative values of maximum flow (i.e., CMF). The maximum flows calculated by the robust model are about 109%, 30%, and 13% greater than the maximum flows calculated by the deterministic model on average for the 50×50 , 100×100 , and 200×200 test networks, respectively. In contrast, the actual maximum flows achieved by the robust model on the 50×50 , 100×100 , and 200×200 test networks are lower than the deterministic model by 47.9%, 21.5%, and 9.7% on average, respectively. Figure 5.9 graphically represents the ratio calculated maximum flow (CMF) to the actual maximum flow (AMF), and the percentage reduction in actual maximum flow observed while using the

robust model instead of the deterministic model. It can be seen that the ratio of CMF to AMF significantly higher for the robust model and is also closer to 100% than for the deterministic model, indicating that the robust model is more reliable than the deterministic model.

		п		50×50			100×100)		200×200)	500×500
		11	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	500×500
Calculated	0	0	3742	4040	4227	9350	9720	9921	20761	21298	21520	NA
maximum flow	20	2	4096	4378	4566	9758	10157	10404	21253	21802	22019	57231
(units)	20	10	4152	4480	4678	9914	10289	10489	21380	21929	22151	57329
Interdiction	0	0	50.45	51.15	51.62	50.93	51.44	51.87	49.4	51.33	54.17	NA
success rate	20	2	55.69	56.59	57.09	55.76	56.56	57.23	54.71	56.59	58.53	56.89
(%)	20	10	78.53	78.87	79.98	78.04	78.7	80.46	77.67	78.6	79.47	78.22
Miniman	0	0	4033	4265	4484	9549	9890	10114	20890	21405	21652	NA
$\mathbf{AME} (\mathbf{unit}_{a})$	20	2	3984	4268	4426	9424	9902	10152	20824	21415	21708	56476
AMI ^(units)	20	10	3872	4151	4320	9410	9805	9986	20698	21262	21526	56310
Amonomo	0	0	4540	4837	5044	10191	10554	10776	21617	22073	22303	NA
Average	20	2	4504	4786	4938	10140	10508	10731	21565	22025	22265	57048
AMI ^(units)	20	10	4198	4508	4692	9852	10222	10442	21253	21738	21967	56768
Marinaura	0	0	5150	5421	5635	10762	11189	11434	22150	22673	22974	NA
ME(unita)	20	2	4993	5330	5511	10762	11104	11345	22186	22590	22896	57562
AMF (units)	20	10	4553	4871	5034	10253	10655	10886	21695	22186	22480	57333

Table 5.6: Value of adding robustness in decision-making ($\Delta = 2000$)

(a) AMF: Actual Maximum flow found in Monte Carlo Simulation.

(b) Interdiction success rates and flow values are rounded to the nearest hundredths and ones, respectively.

			T 50×50				100×100)		200×200)
	1	11	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max
Calculated Maximum	0	0	607	651	675	4598	4715	4844	14793	15147	15416
flow (units)	60	60	1301	1361	1450	5859	6126	6338	16571	17088	17387
Interdiction success	0	0	51.05	51.33	51.78	50.79	51.22	51.48	50.96	51.61	52.16
(%)	60	60	84.97	85.37	85.72	85.01	85.31	85.59	84.78	85.17	85.83
Minimum increase in	0	0	2324	2514	2659	6526	6994	7381	17264	17871	18160
AMF (%)	60	60	1256	1363	1515	5628	5925	6167	16161	16624	16870
Average increase in	0	0	3268	3468	3672	7937	8219	8396	18675	19088	19352
AMF (%)	60	60	1668	1807	1924	6158	6450	6653	16791	17245	17522
Maximum increase in	0	0	4108	4423	4810	8886	9364	9496	19712	20170	20553
AMF (%)	60	60	2070	2254	2431	6728	7065	7307	17455	17852	18253

Table 5.7: Value of adding robustness in decision-making ($\Delta = 10000$)

(a) AMF: Actual Maximum flow found in Monte Carlo Simulation.

(b) Increases are with respect to the calculated maximum flow

(c) All percentage values are rounded to the nearest hundredths



Figure 5.9: Value of adding robustness ($\Delta = 10000$)

Table 5.8 summarizes the results obtained using the value of parameter Δ as 15000. The Δ value of 15000 represents an interdiction attempt on about 100%, 68%, and 34% of the forward arcs of the minimum cut of 50×50, 100×100, and 200×200 test networks in the deterministic scenario, respectively. A difference in the probability of interdiction success of about 38% across all network sizes between the robust model and the deterministic model is achieved. The robust model provides highly conservative values of maximum flow (i.e., CMF). The maximum flows calculated by the robust model are about 109%, 30%, and 13% greater than the maximum flows calculated by the deterministic model on average for the 50×50, 100×100, and 200×200 test networks, respectively. In contrast, the actual maximum flows achieved by the robust model on the 50×50, 100×100, and 200×200 test networks are lower than the deterministic model by 78.13%, 38.74%, and 16.65% on average, respectively. It is interesting to note that the AMFs obtained by the deterministic model here is greater than the AMFs obtained using the Δ parameter as 10000. The reason for this is that the abundance of interdiction resources actually led to a minimum cut with a larger number of forward arcs, leading to higher actual maximum flows. Figure 5.10 graphically represents the ratio calculated maximum flow (CMF) to the actual maximum flow (AMF), and the percentage reduction in actual maximum flow observed while using the robust model instead of the deterministic model. It can be seen that the ratio of CMF to AMF here is lower for the deterministic scenario here than for the case when Δ is 10000 (see Figure 5.9). However, for the robust model the ratio becomes less reliable only for the 50×50 test network, and remains relatively stable for the 100×100 and 200×200 test networks.



Figure 5.10: Value of adding robustness ($\Delta = 15000$)

			50×50 100×100)	200×200			
	1	11	Min	Ave	Max	Min	Ave	Max	Min	Ave	Max	
Calculated Maximum	0	0	0	0	0	2635	2719	2791	11868	12072	12235	
flow (units)	60	100	300	359	412	4209	4336	4484	14415	14815	15052	
Interdiction success	0	0	51.16	51.33	51.52	51.05	51.28	51.48	50.88	51.51	52.1	
(%)	60	100	89.39	89.67	89.88	89.24	89.48	89.83	89.08	89.49	89.85	
Minimum increase in	0	0	2402	2683	2960	5738	6126	6521	15953	16332	16750	
AMF (%)	60	100	400	488	543	3959	4074	4238	13892	14140	14419	
Average increase in	0	0	3559	3813	4144	7191	7477	7687	17302	17655	17938	
AMF (%)	60	100	766	834	877	4437	4580	4730	14336	14715	14960	
Maximum increase in	0	0	4659	4917	5434	8319	8738	9251	18475	18863	19129	
AMF (%)	60	100	1061	1193	1281	4903	5129	5371	14848	15280	15626	

Table 5.8: Value of adding robustness in decision-making ($\Delta = 15000$)

(a) AMF: Actual Maximum flow found in Monte Carlo Simulation.

(b) Increases are with respect to the calculated maximum flow

(c) All percentage values are rounded to the nearest hundredths

5.5 Summary

In this chapter, computational analyses are performed on various test networks. Firstly, the computational efficiency of heuristics is established and compared to solving it using a state-of-the-art MIP solver. The MIP solver provides the best solution quality as well as confidence in the solution, but the computational time requirement is unreasonably large. The Lagrangian Relaxation heuristic (LR) provides very good solution quality very quickly, but the confidence in the solution is lacking. A heuristic-based on Benders' Decomposition (BD) is developed which provides better solution confidence bounds than LR but lacks the solution quality of LR. Finally, an Enhanced Benders' Decomposition based heuristic (EBD) is developed which merges the above LR and BD heuristics. EBD achieves solution quality of LR with solution confidence bounds better than BD, in a very short computation time.

The above analysis is followed by a sensitivity study on how the amount of uncertainty in the arc capacity and resource consumption affects the solution quality. A Monte Carlo simulation scheme is developed to simulate actual networks and test the robustness of the decision at three levels of uncertainty in network parameter estimation: Low, Moderate, and High. It was found that the computed maximum flows slightly overestimated the actual maximum flows for the Moderate and High uncertainty cases, but significantly underestimated the actual maximum flows for the Low uncertainty cases. As the test networks with Low level of uncertainty in network parameters provided the most challenge, further analyses were performed only for them. To improve the underestimation of actual maximum flow, the budgets of uncertainty are increased. The sensitivity to the changes in the budgets of uncertainty was performed by changing only one of the Γ or Π parameters from the base case, to check their effect. The changes in the parameter Γ only affect the calculated maximum flow for the network, without changing the interdiction success rates and actual maximum flows. Whereas, an increase in the parameter Π increases the interdiction success rate and reduces the actual maximum flows. The Γ , Π combination of 20,10 was found to provide the best results for how the randomness and the success of an interdiction attempt were described in the Monte Carlo scheme.

The final analysis of the value of considering robustness showed that the interdiction decisions vary in even the trivial small networks. The analysis of the interdiction success rates and the actual maximum flows in case of a deterministic decision and a robust decision were compared. It was found that though the estimates of the robust decision are more conservative than the deterministic decision, the reliability of the solution as well as the final outcomes are superior for the robust decisions (i.e. the robust decision provides a better estimate of actual maximum flow, and the actual maximum flow is significantly lower for the robust case than for the deterministic case). When the interdiction attempts are larger, the robust solutions outperform the deterministic solutions with a practically significant improvement in reduction of maximum flows as well as the reliability of maximum flow estimates provided by the model. The next chapter concludes this thesis and suggests some directions for future research.

6 Conclusions

6.1 Summary and Conclusions

Network-based structures are ubiquitous, and therefore, the identification and study of the vulnerable connections of the network are imperative. The overarching goal of this thesis was the identification of the vulnerable connections of the network under uncertainty. Specifically, the identification of the vulnerable arcs of a maximum flow network was undertaken using a network interdiction framework considering uncertainty in arc capacity and interdiction resource consumption. In this study, network interdiction consists of two players, the adversary and the interdictor, each of whom gets a single turn after which the game ends. The goal of the adversary is to maximize the flow through the network and moves second in the game. The goal of the interdictor is to interdict arcs to minimize the maximum flow that can be achieved by the adversary and moves first in the game. The uncertainty is incorporated into the model using robust optimization with polyhedral uncertainty sets, introduced by Bertsimas and Sim (2004). The problem is initially formulated as a bilevel program, each level representing the decisionmaking undertaken by each player. As the bilevel formulation can not be solved directly, it is reformulated as a mixed-integer linear program. This mixed-integer linear formulation represents the maximum flow network interdiction model considering uncertainties in arc capacity and resource consumption, abbreviated as Robust Network Interdiction Problem (RNIP). When the budgets of uncertainty are set to 0, RNIP reduces to the deterministic maximum flow network interdiction model proposed by Wood (1993). Extensions of RNIP, one requiring multiple resources for interdiction, and the other allowing discrete partial interdiction are proposed. However, solving these extensions is not a part of this thesis.

RNIP is an NP-Hard problem to solve, and therefore, solving it exactly for moderate- and large-sized networks can be very time-consuming. In some real-life scenarios, the information regarding the adversary may be time-sensitive requiring quick decision-making [Kenney (2003), Meyer and Anderson (2008)]. This motivates the development of three efficient heuristics. The first heuristic is based on Lagrangian Relaxation (LR). Valid upper bounds of the unbounded variables in RNIP are derived to prevent unboundedness in LR sub-problems. Maximum flow - minimum cut strong duality and robust knapsack problem [Bertsimas and Sim (2003)], along with the Lagrangian parameters, are used to derive the upper bound solution of RNIP. The second heuristic is based on Benders' Decomposition (BD). RNIP is partitioned in such a manner that the BD sub-problem only results in optimality cuts, ensuring that lower bound and upper bound solutions can be obtained after every iteration. The BD master problem is time-expensive to be solved exactly using an MIP solver as its inherent complexity is the same as that of RNIP. A simultaneous penalty heuristic is designed to efficiently solve

the BD master problem. It was found that LR led to a very strong upper solution, but a weak lower bound solution and BD led to moderate lower and upper bound solutions to RNIP. The third heuristic aims to merge the best attributes of LR and BD into one and is called Enhanced Benders' Decomposition (EBD). EBD is achieved by first solving the problem using LR, and then using the final upper bound solution of LR to derive the first optimality cut for BD. The final bounds for LR also act as valid initial bounds for BD in the EBD heuristic. EBD achieves the upper bound of LR and stronger lower bounds than BD for RNIP, making it a true amalgam of LR and BD.

All the three developed heuristics are very time efficient, achieving better than 85% time savings compared to a state-of-the-art MIP solver across all the 31 test networks considered in the study. Among the three heuristics, EBD provides the best trade-off between computational time-efficiency and solution confidence (lesser than 5% MIP gap across all test networks). A recent survey of network interdiction models and algorithms by Smith and Song (2019) cites computational efficiency as a major bottleneck faced by adaptive defense strategies, especially in large networks. The EBD heuristic proposed here can get the first solution in about 11 minutes with a gap of 1.1% for a network consisting of 250,000 nodes and 998,000 directed arcs, in contrast to the state-of-the-art MIP solver which fails to compute an initial solution in 12 hours. Sensitivity analysis is performed to identify the best performing combination of budgets of robustness using a Monte-Carlo simulation scheme. A robust decision provides a much better estimate as well as a reduction in the maximum flows for the scenarios generated by the Monte Carlo scheme, compared to a decision not considering uncertainty. In the present study, RNIP provides estimates which are less than 6% from maximum flows found in Monte Carlo simulations compared to at least 26% underestimation when no uncertainties are considered, on average for 100×100 test networks and larger. The robust decisions also lead to a reduction in average actual maximum flows in the simulations, about 78% reduction for 50×50 test networks, 39% reduction for 100×100 test networks, and 17% reduction for 200×200 test networks, when 15000 units of resources are spent on interdiction attempt compared to cases of no uncertainty consideration. This shows that considering uncertainty in decision-making results in more reliable predictions as well as leads to practically significant improvement in the objective, here, minimization of maximum flow in a network.

6.2 Directions for future research

Several assumptions are made in this work (refer section 3.2 for the compilation of all the assumptions). While most of the assumptions are readily generalizable, the others can be tackled to make this work more widely applicable. First is that the adversary has no information about the interdiction attempt made by the interdictor. This assumption is necessary to model a static network that is used by both the players. However, like the interdictor has information about the adversary, the adversary too might gain information about the interdiction attempt. This could lead to changes in the network structure after the interdictor's turn like the appearance of new nodes and arcs, and abandonment of some old nodes and arcs. Dynamic network modeling in the context of network interdiction under uncertainty could be a possible future direction.

The second assumption is that the uncertainty considered in modeling as well as during analysis is symmetrical about the nominal value considered. However, this may not always be true. Application of distributionally robust optimization to network interdiction problems could be explored.

The third assumption is that the adversary knows the source and the destination for the adversary's commodity. New frameworks for considering incomplete information in uncertain network interdiction models can be considered.

Currently, the problem models decision-making by the interdictor on the adversary's network. The study can be extended to model decision-making by the defender on its network, which is attacked by the adversary. Even though the heuristics proposed here are very computationally efficient and provide great confidence in the solution, they do not provide the exact solution. The development of exact time-efficient solution algorithms could be looked into.

Bibliography

- 2018 National Drug Threat Assessment, 2018. Strategic Intelligence Section, Drug Enforcement Administration, United States Department of Justice, publication number: DEA-DCT-DIR-032-18.
- World Drug Report 2018, 2018. URL https://www.unodc.org/wdr2018. United Nations Office on Drugs and Crime, publication number: E.18.XI.9.
- M. Afshari Rad and H. T. Kakhki. Two extended formulations for cardinality maximum flow network interdiction problem. *Networks*, 69(4):367–377, 2017.
- R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. Network Flows: Theory, Algorithms, and Applications. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1993. ISBN 0-13-617549-X.
- I. Akgün, B. Ç. Tansel, and R. K. Wood. The multi-terminal maximum-flow network-interdiction problem. *European Journal of Operational Research*, 211 (2):241–251, 2011.
- D. S. Altner, Ö. Ergun, and N. A. Uhan. The maximum flow network interdiction problem: valid inequalities, integrality gaps, and approximability. *Operations Research Letters*, 38(1):33–38, 2010.
- T. Arentze and H. Timmermans. Information gain, novelty seeking and travel: a model of dynamic activity-travel behavior under conditions of uncertainty. *Transportation Research Part A: Policy and Practice*, 39(2-3):125–145, 2005.
- J. M. Arroyo. Bilevel programming applied to power system vulnerability analysis under multiple contingencies. *IET generation, transmission & distribution*, 4(2): 178–190, 2010.
- N. Assimakopoulos. A network interdiction model for hospital infection control. Computers in biology and medicine, 17(6):413–422, 1987.

- A. Atamtürk, C. Deck, and H. Jeon. Successive quadratic upper-bounding for discrete mean-risk minimization and network interdiction. *INFORMS Journal* on Computing, 2019.
- E. Ayyildiz, G. Ozçelik, and C. T. Gencer. Determining the most vital arcs on the shortest path for fire trucks in terrorist actions that will cause fire. *Communications Series A1 Mathematics & Statistics*, 68(1), 2018.
- H. Bayrak and M. D. Bailey. Shortest path network interdiction with asymmetric information. *Networks: An International Journal*, 52(3):133–140, 2008.
- A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. Operations research letters, 25(1):1–13, 1999.
- A. Ben-Tal and A. Nemirovski. Robust optimization-methodology and applications. *Mathematical Programming*, 92(3):453–480, 2002.
- A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust optimization*, volume 28. Princeton University Press, 2009.
- J. F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische mathematik*, 4(1):238–252, 1962.
- D. Bertsimas and M. Sim. Robust discrete optimization and network flows. *Mathematical programming*, 98(1-3):49–71, 2003.
- D. Bertsimas and M. Sim. The price of robustness. *Operations research*, 52(1): 35–53, 2004.
- D. Bertsimas, D. B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM review*, 53(3):464–501, 2011.
- D. Bertsimas, E. Nasrabadi, and S. Stiller. Robust and adaptive network flows. *Operations Research*, 61(5):1218–1242, 2013.
- D. Bertsimas, E. Nasrabadi, and J. B. Orlin. On the power of randomization in network interdiction. *Operations Research Letters*, 44(1):114–120, 2016.
- L. Bingol. A Lagrangian heuristic for solving a network interdiction problem. Master's thesis, 2001. Naval Postgraduate School Monterey CA.

- V. L. Boginski, C. W. Commander, and T. Turko. Polynomial-time identification of robust network flows under uncertain arc failures. *Optimization Letters*, 3(3): 461–473, 2009.
- R. Borndörfer, G. Sagnol, and S. Schwartz. An extended network interdiction problem for optimal toll control. *Electronic Notes in Discrete Mathematics*, 52: 301–308, 2016.
- J. S. Borrero, O. A. Prokopyev, and D. Sauré. Sequential shortest path interdiction with incomplete information. *Decision Analysis*, 13(1):68–98, 2015.
- G. Brown, M. Carlyle, J. Salmerón, and K. Wood. Defending critical infrastructure. *Interfaces*, 36(6):530–544, 2006.
- P. Cappanera and M. P. Scaparra. Optimal allocation of protective resources in shortest-path networks. *Transportation Science*, 45(1):64–80, 2011.
- D. Chaerani and C. Roos. Modelling some robust design problems via conic optimization. In Operations Research Proceedings 2006, pages 209–214. Springer, 2007.
- A. Charnes, W. W. Cooper, and G. H. Symonds. Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil. *Management* science, 4(3):235–263, 1958.
- R. L. Church and M. P. Scaparra. Protecting critical assets: the r-interdiction median problem with fortification. *Geographical Analysis*, 39(2):129–146, 2007.
- R. L. Church, M. P. Scaparra, and R. S. Middleton. Identifying critical infrastructure: the median and covering facility interdiction problems. Annals of the Association of American Geographers, 94(3):491–502, 2004.
- K. J. Cormican. Computational methods for deterministic and stochastic network interdiction problems. Master's thesis, 1995. Naval Postgraduate School Monterey CA.
- K. J. Cormican, D. P. Morton, and R. K. Wood. Stochastic network interdiction. Operations Research, 46(2):184–197, 1998.
- G. B. Dantzig. Linear programming under uncertainty. Management Science, 1 (3,4):197–206, 1955.
- E. Delage and Y. Ye. Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations research*, 58(3): 595–612, 2010.
- E. Dews and F. Kozaczka. Air interdiction: Lessons from past campaigns. Technical report, Rand Corporation Santa Monica CA, 1981.
- T. Ding, L. Yao, and F. Li. A multi-uncertainty-set based two-stage robust optimization to defender-attacker-defender model for power system protection. *Reliability Engineering & System Safety*, 169:179–186, 2018.
- F. Enayaty-Ahangar, C. E. Rainwater, and T. C. Sharkey. A logic-based decomposition approach for multi-period network interdiction models. *Omega*, 87: 71–85, 2019.
- M. L. Fisher. An applications oriented guide to lagrangian relaxation. *Interfaces*, 15(2):10–21, 1985.
- X. Fu and E. Modiano. Network interdiction using adversarial traffic flows. In IEEE INFOCOM 2019-IEEE Conference on Computer Communications, pages 1765–1773. IEEE, 2019.
- V. Gabrel, C. Murat, and A. Thiele. Recent advances in robust optimization: An overview. *European journal of operational research*, 235(3):471–483, 2014.
- T. P. Gannon. The dual decomposition method and its application to an interdicted network. Master's thesis, 1989. Naval Postgraduate School Monterey CA.
- R. Gedik, H. Medal, C. Rainwater, E. A. Pohl, and S. J. Mason. Vulnerability assessment and re-routing of freight trains under disruptions: A coal supply chain network application. *Transportation Research Part E: Logistics and Transportation Review*, 71:45–57, 2014.
- J. Goh and M. Sim. Distributionally robust optimization and its tractable approximations. Operations research, 58(4-part-1):902–917, 2010.

- A. V. Goldberg and R. E. Tarjan. A new approach to the maximum-flow problem. Journal of the ACM (JACM), 35(4):921–940, 1988.
- D. Granata, G. Steeger, and S. Rebennack. Network interdiction via a critical disruption path: branch-and-price algorithms. *Computers & Operations Research*, 40(11):2689–2702, 2013.
- H. J. Greenberg and T. Morrison. Robust optimization. CRC Press, 2008.
- L. Gurobi Optimization. Gurobi optimizer reference manual, 2019. URL http: //www.gurobi.com.
- A. Hagberg, P. Swart, and D. S Chult. Exploring network structure, dynamics, and function using NetworkX. In G. Varoquaux, T. Vaught, and J. Millman, editors, *Proceedings of the 7th Python in Science Conference (SciPy2008)*, pages 11–15, 2008.
- S. Han, Z. Peng, and S. Wang. The maximum flow problem of uncertain network. Information Sciences, 265:167–175, 2014.
- E. Israeli and R. K. Wood. Shortest-path network interdiction. Networks: An International Journal, 40(2):97–111, 2002.
- P. Kall, S. W. Wallace, and P. Kall. *Stochastic programming*, volume 2. Springer, 1994.
- M. C. Kenney. Intelligence games: Comparing the intelligence capabilities of law enforcement agencies and drug trafficking enterprises. *International Journal of Intelligence and CounterIntelligence*, 16(2):212–243, 2003.
- A. Kheirkhah, H. Navidi, and M. Messi Bidgoli. A bi-level network interdiction model for solving the hazmat routing problem. *International Journal of Produc*tion Research, 54(2):459–471, 2016.
- W. H. Lam, H. Shao, and A. Sumalee. Modeling impacts of adverse weather conditions on a road network with uncertainties in demand and supply. *Trans*portation research part B: methodological, 42(10):890–910, 2008.
- L. S. Lasdon. Optimization Theory for Large Systems. Dover Publications, Inc., Mineola, New York, 2002.

- C. Lee, K. Lee, K. Park, and S. Park. Branch-and-price-and-cut approach to the robust network design problem without flow bifurcations. *Operations Research*, 60(3):604–610, 2012.
- T. L. Lei. Identifying critical facilities in hub-and-spoke networks: A hub interdiction median problem. *Geographical Analysis*, 45(2):105–122, 2013.
- C. Lim and J. C. Smith. Algorithms for discrete and continuous multicommodity flow network interdiction problems. *IIE Transactions*, 39(1):15–26, 2007.
- K.-H. Loh. Modelling strategic strikes against transportation networks. Master's thesis, 1991. Naval Postgraduate School Monterey CA.
- Q.-C. Lu, Z.-R. Peng, and J. Zhang. Identification and prioritization of critical transportation infrastructure: Case study of coastal flooding. *Journal of Transportation Engineering*, 141(3):04014082, 2014.
- B. J. Lunday and H. D. Sherali. A dynamic network interdiction problem. *Infor*matica, 21(4):553–574, 2010.
- N. R. Magliocca, K. McSweeney, S. E. Sesnie, E. Tellman, J. A. Devine, E. A. Nielsen, Z. Pearson, and D. J. Wrathall. Modeling cocaine traffickers and counterdrug interdiction forces as a complex adaptive system. *Proceedings of the National Academy of Sciences*, 116(16):7784–7792, 2019.
- A. Malaviya, C. Rainwater, and T. Sharkey. Multi-period network interdiction problems with applications to city-level drug enforcement. *IIE Transactions*, 44 (5):368–380, 2012.
- H. Markowitz. Portfolio selection. The journal of finance, 7(1):77–91, 1952.
- H. Markowitz. Portfolio selection: Efficient Diversification of Investments. Number 2nd. John Wiley & Sons, 1991.
- A. W. McMasters and T. M. Mustin. Optimal interdiction of a supply network. Naval Research Logistics Quarterly, 17(3):261–268, 1970.
- A. D. Meyer and D. P. Anderson. The air national guard and the war on drugs: non-state actors before 9/11. Air Power History, 55(3):12–29, 2008.

- D. P. Michalopoulos, J. W. Barnes, and D. P. Morton. Prioritized interdiction of nuclear smuggling via tabu search. *Optimization Letters*, 9(8):1477–1494, 2015.
- M. Minoux. On robust maximum flow with polyhedral uncertainty sets. Optimization Letters, 3(3):367–376, 2009.
- M. Minoux. Robust network optimization under polyhedral demand uncertainty is np-hard. *Discrete Applied Mathematics*, 158(5):597–603, 2010.
- M. Monaci, U. Pferschy, and P. Serafini. Exact solution of the robust knapsack problem. *Computers & operations research*, 40(11):2625–2631, 2013.
- D. P. Morton, F. Pan, and K. J. Saeger. Models for nuclear smuggling interdiction. *IIE Transactions*, 39(1):3–14, 2007.
- A. L. Motto, J. M. Arroyo, and F. D. Galiana. A mixed-integer lp procedure for the analysis of electric grid security under disruptive threat. *IEEE Transactions* on Power Systems, 20(3):1357–1365, 2005.
- A. K. Nandi and H. R. Medal. Methods for removing links in a network to minimize the spread of infections. *Computers & Operations Research*, 69:10–24, 2016.
- M. V. Nehme and D. P. Morton. Tightening a network interdiction model. In *IIE Annual Conference. Proceedings*, page 1304. Institute of Industrial and Systems Engineers (IISE), 2009.
- A. Nemirovski and A. Shapiro. Convex approximations of chance constrained programs. *SIAM Journal on Optimization*, 17(4):969–996, 2006.
- S. Neumayer, A. Efrat, and E. Modiano. Geographic max-flow and min-cut under a circular disk failure model. *Computer Networks*, 77:117–127, 2015.
- F. Ordóñez and J. Zhao. Robust capacity expansion of network flows. *Networks:* An International Journal, 50(2):136–145, 2007.
- F. Pan and D. P. Morton. Minimizing a stochastic maximum-reliability path. Networks: An International Journal, 52(3):111–119, 2008.

- F. Pan, W. S. Charlton, and D. P. Morton. A stochastic program for interdicting smuggled nuclear material. In *Network interdiction and stochastic integer* programming, pages 1–19. Springer, 2003.
- L. Perelman and S. Amin. A network interdiction model for analyzing the vulnerability of water distribution systems. In *Proceedings of the 3rd international* conference on High confidence networked systems, pages 135–144. ACM, 2014.
- C. A. Phillips. The network inhibition problem. In *Proceedings of the twenty-fifth* annual ACM symposium on Theory of computing, pages 776–785. ACM, 1993.
- M. A. Rad and H. T. Kakhki. Maximum dynamic network flow interdiction problem: New formulation and solution procedures. *Computers & Industrial Engineering*, 65(4):531–536, 2013.
- R. Rahmaniani, T. G. Crainic, M. Gendreau, and W. Rei. The benders decomposition algorithm: A literature review. *European Journal of Operational Re*search, 259(3):801–817, 2017.
- H. D. Ratliff, G. T. Sicilia, and S. Lubore. Finding the n most vital links in flow networks. *Management Science*, 21(5):531–539, 1975.
- C. M. Rocco S. and J. E. Ramirez-Marquez. Deterministic network interdiction optimization via an evolutionary approach. *Reliability Engineering System* Safety, 94(2):568 – 576, 2009.
- C. M. Rocco S. and J. E. Ramirez-Marquez. A bi-objective approach for shortestpath network interdiction. *Computers & Industrial Engineering*, 59(2):232–240, 2010.
- J. Rosenhead. Robustness analysis: Keeping your options open. Number 2nd. John Wiley and Sons, 2001.
- J. O. Royset and R. K. Wood. Solving the bi-objective maximum-flow networkinterdiction problem. *INFORMS Journal on Computing*, 19(2):175–184, 2007.
- S. Sadeghi, A. Seifi, and E. Azizi. Trilevel shortest path network interdiction with partial fortification. *Computers & Industrial Engineering*, 106:400–411, 2017.

- J. Salmeron, K. Wood, and R. Baldick. Analysis of electric grid security under terrorist threat. *IEEE Transactions on power systems*, 19(2):905–912, 2004.
- J. Salmeron, K. Wood, and R. Baldick. Worst-case interdiction analysis of largescale electric power grids. *IEEE Transactions on power systems*, 24(1):96–104, 2009.
- A. Sanjab, W. Saad, and T. Başar. Prospect theory for enhanced cyber-physical security of drone delivery systems: A network interdiction game. In 2017 IEEE International Conference on Communications (ICC), pages 1–6. IEEE, 2017.
- M. P. Scaparra and R. L. Church. A bilevel mixed-integer program for critical infrastructure protection planning. *Computers & Operations Research*, 35(6): 1905–1923, 2008.
- J. A. Sefair and J. C. Smith. Dynamic shortest-path interdiction. *Networks*, 68(4): 315–330, 2016.
- A. Shapiro, D. Dentcheva, and A. Ruszczyński. Lectures on stochastic programming: modeling and theory. SIAM, 2009.
- J. C. Smith and C. Lim. Algorithms for network interdiction and fortification games. In *Pareto optimality, game theory and equilibria*, pages 609–644. Springer, 2008.
- J. C. Smith and Y. Song. A survey of network interdiction models and algorithms. European Journal of Operational Research, 2019.
- Y. Song and S. Shen. Risk-averse shortest path interdiction. INFORMS Journal on Computing, 28(3):527–539, 2016.
- A. L. Soyster. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations research*, 21(5):1154–1157, 1973.
- R. L. Steinrauf. Network interdiction models. Master's thesis, 1991. Naval Postgraduate School Monterey CA.
- K. M. Sullivan, D. P. Morton, F. Pan, and J. Cole Smith. Securing a border under asymmetric information. *Naval Research Logistics (NRL)*, 61(2):91–100, 2014.

- A. Sumalee, K. Uchida, and W. H. Lam. Stochastic multi-modal transport network under demand uncertainties and adverse weather condition. *Transporta*tion Research Part C: Emerging Technologies, 19(2):338–350, 2011.
- E. Towle and J. Luedtke. New solution approaches for the maximum-reliability stochastic network interdiction problem. *Computational Management Science*, 15(3-4):455–477, 2018.
- A. Uygun. Network interdiction by lagrandian relaxation and branch-and-bound. Master's thesis, 2002. Naval Postgraduate School Monterey CA.
- X. Wei, C. Zhu, K. Xiao, Q. Yin, and Y. Zha. Shortest path network interdiction with goal threshold. *IEEE Access*, 6:29332–29343, 2018.
- W. Wiesemann, D. Kuhn, and M. Sim. Distributionally robust convex optimization. Operations Research, 62(6):1358–1376, 2014.
- R. Wollmer. Removing arcs from a network. Operations Research, 12(6):934–940, 1964.
- R. K. Wood. Deterministic network interdiction. Mathematical and Computer Modelling, 17(2):1–18, 1993.
- R. K. Wood. Bilevel network interdiction models: Formulations and solutions. Wiley encyclopedia of operations research and management science, 2010.
- K. Xu, K. Xiao, Q. Yin, Y. Zha, and C. Zhu. Bridging the gap between observation and decision making: Goal recognition and flexible resource allocation in dynamic network interdiction. In *IJCAI*, pages 4477–4483, 2017.
- J. Yates. Network interdiction methods and approximations in a hazmat transportation setting. In Handbook of OR/MS Models in Hazardous Materials Transportation, pages 187–243. Springer, 2013.
- J. Yates and K. Lakshmanan. A constrained binary knapsack approximation for shortest path network interdiction. *Computers & Industrial Engineering*, 61(4): 981–992, 2011.

- W. Yuan, J. Wang, F. Qiu, C. Chen, C. Kang, and B. Zeng. Robust optimizationbased resilient distribution network planning against natural disasters. *IEEE Transactions on Smart Grid*, 7(6):2817–2826, 2016.
- J. Zhang, J. Zhuang, and B. Behlendorf. Stochastic shortest path network interdiction with a case study of arizona-mexico border. *Reliability Engineering &* System Safety, 179:62–73, 2018.
- J. Zheng and D. A. Castañón. Dynamic network interdiction games with imperfect information and deception. In 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), pages 7758–7763. IEEE, 2012a.
- J. Zheng and D. A. Castanón. Stochastic dynamic network interdiction games. In 2012 American Control Conference (ACC), pages 1838–1844. IEEE, 2012b.